

Asymptotes in $SU(2)$ Recoupling Theory: Wigner Matrices, $3j$ Symbols, and Character Localization

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Abstract

In this paper we employ a novel technique combining the Euler Maclaurin formula with the saddle point approximation method to obtain the asymptotic behavior (in the limit of large representation index J) of generic Wigner matrix elements $D_{MM'}^J(g)$. We use this result to derive asymptotic formulae for the character $\chi^J(g)$ of an $SU(2)$ group element and for Wigner's $3j$ symbol. Surprisingly, given that we perform five successive layers of approximations, the asymptotic formula we obtain for $\chi^J(g)$ is in fact *exact*. This result provides a non trivial example of a Duistermaat-Heckman like localization property for discrete sums.

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1 Introduction

The saddle point approximation (SPA) is a classical algorithm to determine asymptotic behavior of a large class of integrals in some large parameter limit [1]. One uses it when exact calculations are either too complex or not very relevant. Recently SPA has been used in conjunction with the Euler Maclaurin (EM) formula to derive asymptotic behavior of discrete sums [2, 3]. In the combined EM SPA scheme corrections to the leading behavior come from two sources: the derivative terms in the EM formula and sub leading terms in the SPA estimate.

In this paper we use the EM SPA method to derive the asymptotic behavior of Wigner rotation matrix elements. We subsequently use this asymptotic formula to derive the asymptotic behavior of the character of an $SU(2)$ group element. Although our estimate is obtained after using twice the EM SPA approximation and once the Stirling approximation for Euler's Gamma functions it turns out to be the exact result. We then proceed to obtain the asymptotic expression for Wigner's $3j$ symbol, recovering with this method the results of [4].

Our results are relevant for computing topological (Turaev Viro like [5]) invariants and in connection to the volume conjecture [6]. From a theoretical physics perspective they are of consequence for spin foam models [7], Group Field Theory [8, 9], discretized BF theory and lattice gravity [10],[11], [12]. Continuous SPA has been extensively used in this context to derive asymptotic behaviors of spin foam amplitudes [13], [14], [15], and [16], [17], [18].

In the recoupling theory of $SU(2)$, the EM SPA method has already been used to obtain in a particularly simple way the Ponzano-Regge asymptotic of the $6j$ symbol [3], [19]. The main strength of this approach is the following. Most relevant quantities in the recoupling theory of $SU(2)$ are expressed in Fourier space by discrete sums. In particular, the Wigner matrix elements admit a single sum representation [20]. However, generically, the sums are alternated hence difficult to handle. Our EM SPA method deals very efficiently with alternating signs: generically such signs lead to complex saddle points situated outside the initial summation interval. After exchanging the original sums (via the EM formula) for integrals one only deforms the integration contour in the complex plane to pass through the saddle points in a completely standard manner. This feature is the crucial strength of our method, and allows rapid access to explicit results. The EM SPA method should allow one to prove for instance the asymptotic behavior [21] of the $9j$ symbol.

The proofs of our three main results (Theorems 1, 2 and 3) are straightforward, but the sheer amount of computations performed renders this a somewhat technical paper. In Section 2 we give a quick review of iterated saddle point approximations. In Section 3 we establish Theorem 1 and use it in Section 4 to derive the character formula (Theorem 2). Section 5 proves the asymptotic formulae of the $3j$ symbol (Theorem 3). Section 6 draws the conclusion of our work and discusses the relation between our result for the character and the Duistermaat Heckman theorem. The (very detailed) Appendices present explicit computations and detail the EM derivative terms.

2 Successive saddle point approximations

We briefly review the iterated SPA approximations. The result of this section justifies the use of our asymptote of the Wigner matrices to derive the asymptotic behavior of $SU(2)$ characters and Wigner $3j$ symbols.

Consider a function f of two real variables. We are interested in evaluating the asymptotic behavior of the integral

$$I = \int du dx e^{Jf(u,x)}, \quad (1)$$

for large J . One can chose to either evaluate I via an SPA in both variables at the same time or via two successive SPA, one for each variable. The question is if the two estimates coincide. This problem is addressed in full detail in [1] and the answer to the above question is yes (for sufficiently smooth functions), with known estimates. Let us give a quick flavor of the origin of this result.

Remark 1. *Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ be a function with and unique critical point (u_c, x_c) and non degenerate Hessian at (u_c, x_c) such that $I = \int e^{Jf(u,x)}$ admits a SPA at large J . Assume that the equation $\partial_u f(u, x) = 0$ admits an unique solution $u_c = h(x)$, such that $[\partial_u^2 f](h(x), x) \neq 0$. Then the SPA of $\int e^{Jf(u,x)}$ in both variables (u, x) gives the same estimate as two successive SPAs, the first one in u and the second one in x .*

Proof: The simultaneous SPA in u and x yields the estimate

$$I \approx \frac{2\pi}{J \sqrt{[\partial_u^2 f \partial_x^2 f - (\partial_u \partial_x f)^2] \Big|_{(u_c, x_c)}}} e^{Jf(u_c, x_c)}. \quad (2)$$

The saddle point equation for u , $[\partial_u f](u, x) = 0$, is solved by $u_c = h(x)$. Thus a first SPA in u yields

$$I \approx \sqrt{\frac{2\pi}{J}} \int dx \frac{1}{\sqrt{-\partial_u^2 f|_{(h(x), x)}}} e^{Jf(h(x), x)}. \quad (3)$$

We evaluate eq. (3) by a second SPA, in the x variable. The saddle point equation is

$$\frac{d}{dx} \left(f(h(x), x) \right) = [\partial_u f] \Big|_{(h(x), x)} \frac{dh}{dx} + [\partial_x f] \Big|_{(h(x), x)}, \quad (4)$$

and, as $[\partial_u f](h(x), x) = 0$, the first term above vanishes. The critical point x_c is therefore solution of $[\partial_x f] \Big|_{(h(x), x)} = 0$. The second derivative of $f(h(x), x)$ computes to

$$\frac{d^2}{dx^2} \left(f(h(x), x) \right) = \frac{d}{dx} \left([\partial_x f] \Big|_{(h(x), x)} \right) = [\partial_u \partial_x f] \Big|_{(h(x), x)} \frac{dh}{dx} + [\partial_x^2 f] \Big|_{(h(x), x)}, \quad (5)$$

and noting that

$$\frac{d}{dx} [\partial_u f] \Big|_{(h(x), x)} = 0 \Rightarrow [\partial_u^2 f] \Big|_{(h(x), x)} \frac{dh}{dx} + \partial_x [\partial_u f] \Big|_{(h(x), x)} = 0 \Rightarrow \frac{dh}{dx} = - \frac{[\partial_x \partial_u f] \Big|_{(h(x), x)}}{[\partial_u^2 f] \Big|_{(h(x), x)}}, \quad (6)$$

the estimate obtained by two successive SPAs is

$$I \approx \frac{2\pi}{J \sqrt{\partial_u^2 f \Big|_{(h(x_c), x_c)} \left(-\frac{[\partial_u \partial_x f]^2}{[\partial_u^2 f]} + \partial_x^2 f \right) \Big|_{(h(x_c), x_c)}}} e^{J f(u_c, x_c)} , \quad (7)$$

identical with eq. (2). □

This remark generalizes [1], for sufficiently smooth functions of more variables with non degenerate critical points. In the sequel we will express the Wigner matrix elements $D_{MM'}^J$ (up to corrections coming from the EM formula) as integrals which we approximate by a first SPA. To compute more involved sums or integrals of products of such matrix elements (the character of a $SU(2)$ group element and the $3j$ symbol) we will substitute the SPA approximation for each $D_{MM'}^J$ and evaluate the resulting expressions by subsequent SPAs.

3 Asymptotic formula of a Wigner matrix element

In this section we prove an asymptotic formula for a Wigner matrix element. Before proceeding let us mention that many of our results write in terms of angles. In order to avoid issues related to the interval of definition of this angles we will always denote them as $\imath\phi = \ln w$ for some complex number w with $|w| = 1$.

Our starting point is the classical expression of $D_{MM'}^J$ in terms of Euler angles (α, β, γ) in $z y z$ order (see [20])

$$D_{MM'}^J(\alpha, \beta, \gamma) = e^{-\imath\alpha M} e^{-\imath\gamma M'} \sum_t (-)^t \frac{\sqrt{(J+M)!(J-M)!(J+M')!(J-M')!}}{(J+M-t)!(J-M'-t)!t!(t-M+M')!} \xi^{2J+M-M'-2t} \eta^{2t-M+M'} , \quad (8)$$

with $\xi = \cos(\beta/2)$, $\eta = \sin(\beta/2)$. The sum is taken over all t such that all factorials have positive argument (hence it has $1 + \min\{J+M, J-M, J+M', J-M'\}$ terms). We call a Wigner matrix generic if its second Euler angle $\beta \notin \mathbb{Z}\pi$ (that is $0 < \xi^2 < 1$). We define the reduced variables $x = \frac{J}{M}$ and $y = \frac{J}{M'}$. A priori the asymptotic behavior we derive below holds in certain region of the parameters x, y and ξ detailed in Appendices E and C.

Theorem 1. *A generic Wigner matrix element in the spin J representation of a $SU(2)$ group element has in the large J limit the asymptotic behavior*

$$D_{xJ, yJ}^J(\alpha, \beta, \gamma) \approx e^{-\imath J\alpha x - \imath J\gamma y} \left(\frac{1}{\pi J \sqrt{\Delta}} \right)^{\frac{1}{2}} \cos \left[\left(J + \frac{1}{2} \right) \phi + xJ\psi - yJ\omega - \frac{\pi}{4} \right] , \quad (9)$$

with

$$\Delta = (1 - \xi^2)(\xi^2 - xy) - \frac{(x - y)^2}{4} \geq 0 , \quad (10)$$

with ϕ, ψ and ω the three angles

$$\begin{aligned} \imath\phi &= \ln \frac{2\xi^2 - 1 - xy + 2\imath\sqrt{\Delta}}{\sqrt{(1-x^2)(1-y^2)}} , & \imath\psi &= \ln \frac{\frac{x+y}{2} - x\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}} , \\ \imath\omega &= \ln \frac{-\frac{x+y}{2} + y\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)}} . \end{aligned} \quad (11)$$

Proof: The proof of Theorem 1 is divided into two steps: first the approximation of eq. (8) by an integral via the EM formula, and second the evaluation of the latter by an SPA.

Step 1: In the large J limit the leading behavior of the Wigner matrix element eq. (8) is

$$D_{xJ,yJ}^J(\alpha, \beta, \gamma) \approx \frac{1}{2\pi} \int du \sqrt{K(x, y, u)} e^{Jf(x, y, u)} , \quad (12)$$

where

$$\begin{aligned} f(x, y, u) &= -\imath\alpha x - \imath\gamma y + \imath\pi u + (2+x-y-2u) \ln \xi + (2u-x+y) \ln \eta \\ &+ \frac{1}{2}(1-x) \ln(1-x) + \frac{1}{2}(1+x) \ln(1+x) + \frac{1}{2}(1-y) \ln(1-y) + \frac{1}{2}(1+y) \ln(1+y) \\ &- (1+x-u) \ln(1+x-u) - (1-y-u) \ln(1-y-u) \\ &- u \ln u - (u-x+y) \ln(u-x+y) , \end{aligned} \quad (13)$$

and

$$K(x, y, u) = \frac{\sqrt{(1-x)(1+x)(1-y)(1+y)}}{(1+x-u)(1-y-u)(u)(u-x+y)} . \quad (14)$$

To prove this we rewrite eq. (8) in terms of Gamma functions

$$\begin{aligned} D_{MM'}^J(\alpha, \beta, \gamma) &= \sum_t F(J, M, M', t) , \\ F(J, M, M', t) &= e^{\imath\pi t} e^{-\imath\alpha M} e^{-\imath\gamma M'} \xi^{2J+M-M'-2t} \eta^{2t-M+M'} \times \\ &\frac{\sqrt{\Gamma(J+M+1)\Gamma(J-M+1)\Gamma(J+M'+1)\Gamma(J-M'+1)}}{\Gamma(J+M-t+1)\Gamma(J-M'-t+1)\Gamma(t+1)\Gamma(t-M+M'+1)} , \end{aligned} \quad (15)$$

and use the Euler-Maclaurin formula

$$\begin{aligned} \sum_{t_{\min}}^{t_{\max}} h(t) &= \int_{t_{\min}}^{t_{\max}} h(t) dt - B_1 [h(t_{\max}) + h(t_{\min})] \\ &+ \sum_k \frac{B_{2k}}{(2k)!} [h^{(2k-1)}(t_{\max}) - h^{(2k-1)}(t_{\min})] , \end{aligned} \quad (16)$$

where B_1, B_{2k} are the Bernoulli numbers¹. To derive our asymptote we only take into account the integral approximation of eq. (15) (the boundary terms are discussed in Appendix E), hence

$$D_{MM'}^J(\alpha, \beta, \gamma) \approx \int dt F(J, M, M', t) . \quad (17)$$

¹Eq. (16) holds for all C^∞ functions $h(t)$, such that the sum over k converges.

We define $u = \frac{t}{J}$ hence $du = \frac{1}{J}dt$ and using the Stirling formula for the Gamma functions (see Appendix A) we get eq. (12).

Step 2: We now proceed to evaluate the integral (12) by an SPA. Some of the computations relevant for this proof are included in Appendix B. Denoting the set of saddle points by \mathcal{C} , the leading asymptotic behavior of a generic Wigner matrix element writes

$$D_{xJ,yJ}^J(\alpha, \beta, \gamma) \approx \frac{1}{\sqrt{2\pi J}} \sum_{u_* \in \mathcal{C}} \frac{\sqrt{K|_{x,y,u_*}}}{\sqrt{(-\partial_u^2 f)|_{x,y,u_*}}} e^{Jf(x,y,u_*)}. \quad (18)$$

Our task is to identify \mathcal{C} and compute $K|_{x,y,u_*}$, $(-\partial_u^2 f)|_{x,y,u_*}$ and $f(x,y,u_*)$.

The set \mathcal{C} . The derivative of f with respect to u is

$$\partial_u f = i\pi - 2\ln \xi + 2\ln \eta + \ln(1+x-u) + \ln(1-y-u) - \ln u - \ln(u-x+y). \quad (19)$$

A straightforward computation shows that the saddle points are the solutions of

$$(1+x-u)(1-y-u)\frac{(1-\xi^2)}{\xi^2} + u(u-x+y) = 0 \quad (20)$$

$$\Leftrightarrow u^2 - u[2(1-\xi^2) + x - y] + (1-\xi^2)(1+x)(1-y) = 0. \quad (21)$$

The region of parameters x, y, ξ for which the discriminant of eq. (21) is positive gives exponentially suppressed matrix elements while the region for which it is zero gives an Airy function estimate. Both cases are detailed in Appendix C.

In the rest of this proof we treat the region in which the discriminant of eq. (21) is negative. We denote by Δ minus the reduced discriminant, that is

$$\Delta = (1-\xi^2)(\xi^2 - xy) - \frac{(x-y)^2}{4} > 0, \quad (22)$$

and the two saddle points, solutions of eq. (21), write

$$u_{\pm} = (1-\xi^2) + \frac{x-y}{2} \pm i\sqrt{\Delta}, \quad (23)$$

thus the set of saddle points is $\mathcal{C} = \{u_+, u_-\}$.

Evaluation of $f(x, y, u_{\pm})$. We rearrange the terms in eq. (13) to write

$$\begin{aligned} f(x, y, u) = & -i\alpha x - i\gamma y + (2+x-y)\ln \xi + (-x+y)\ln \eta \\ & + \frac{1}{2}(1-x)\ln(1-x) + \frac{1}{2}(1+x)\ln(1+x) + \frac{1}{2}(1-y)\ln(1-y) + \frac{1}{2}(1+y)\ln(1+y) \\ & - (1+x)\ln(1+x-u) - (1-y)\ln(1-y-u) - (-x+y)\ln(u-x+y) \\ & + u \ln \left[(-) \frac{1-\xi^2}{\xi^2} \frac{(1+x-u)(1-y-u)}{u(u-x+y)} \right]. \end{aligned} \quad (24)$$

Note that by the saddle point equations the last line in eq. (24) is zero for u_{\pm} . The rest of eq. (24) computes to (see Appendix B.1 for details)

$$f(x, y, u_{\pm}) = -i\alpha x - i\gamma y \pm i\left(\phi + x\psi - y\omega\right), \quad (25)$$

with

$$\begin{aligned} \imath\phi &= \ln \frac{2\xi^2 - 1 - xy + 2\imath\sqrt{\Delta}}{\sqrt{(1-x^2)(1-y^2)}} , \quad \imath\psi = \ln \frac{\frac{x+y}{2} - x\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}} , \\ \imath\omega &= \ln \frac{-\frac{x+y}{2} + y\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)}} . \end{aligned} \quad (26)$$

Second derivative. The derivative of eq. (19) is

$$-\partial_u^2 f(x, y, u) = \frac{1}{1+x-u} + \frac{1}{1-y-u} + \frac{1}{u} + \frac{1}{u-x+y} . \quad (27)$$

At the saddle points a straightforward computation shows that the second derivative is (see Appendix B.2)

$$(-\partial_u^2 f)|_{x,y,u_{\pm}} = \frac{1}{(1-x^2)(1-y^2)\xi^2(1-\xi^2)} \left(4\Delta \pm \imath 2\sqrt{\Delta} [1+xy-2\xi^2] \right) . \quad (28)$$

The prefactor K . The prefactor $K(x, y, u)$ is

$$K = \frac{\sqrt{(1-x^2)(1-y^2)}}{u(1+x-u)(1-y-u)(u-x+y)} , \quad (29)$$

which computes at the saddle points to (see Appendix B.3)

$$K|_{x,y,u_{\pm}} = \frac{-\sqrt{(1-x^2)(1-y^2)} \left(2\xi^2 - 1 - xy \pm 2\imath\sqrt{\Delta} \right)^2}{\xi^2(1-\xi^2)(1-x^2)^2(1-y^2)^2} . \quad (30)$$

Final evaluation. Before collecting all our previous results we first evaluate, using eq. (28) and (30)

$$\begin{aligned} \frac{K|_{x,y,u_{\pm}}}{(-\partial_u^2 f)|_{x,y,u_{\pm}}} &= - \frac{\left(2\xi^2 - 1 - xy \pm 2\imath\sqrt{\Delta} \right)^2}{\sqrt{(1-x^2)(1-y^2)} \left(4\Delta \pm \imath 2\sqrt{\Delta} [1+xy-2\xi^2] \right)} \\ &= \frac{1}{\sqrt{(1-x^2)(1-y^2)} (\pm 2\imath\sqrt{\Delta})} \left(2\xi^2 - 1 - xy \pm 2\imath\sqrt{\Delta} \right) \\ &= \frac{1}{\pm \imath 2\sqrt{\Delta}} \frac{\left(2\xi^2 - 1 - xy \pm 2\imath\sqrt{\Delta} \right)}{\sqrt{(1-x^2)(1-y^2)}} . \end{aligned} \quad (31)$$

Comparing eq. (31) with eq. (11) we conclude that

$$\frac{K|_{x,y,u_{\pm}}}{(-\partial_u^2 f)|_{x,y,u_{\pm}}} = \frac{1}{\pm \imath 2\sqrt{\Delta}} e^{\pm \imath\phi} . \quad (32)$$

Substituting eq. (32) and (25) into eq. (18) we obtain

$$D_{xJ,yJ}^J(\alpha, \beta, \gamma) \approx \frac{1}{\sqrt{2\pi J}} \left(\frac{1}{2\sqrt{\Delta}} \right)^{\frac{1}{2}} e^{-iJ\alpha x - iJ\gamma y} \left(\sqrt{\frac{1}{i}} e^{i\phi} e^{iJ(\phi + x\psi - y\omega)} + \sqrt{\frac{1}{-i}} e^{-i\phi} e^{-iJ(\phi + x\psi - y\omega)} \right), \quad (33)$$

and a straightforward computation proves Theorem 1. □

4 Characters

In this section we use Theorem 1 to derive an asymptotic formula for the character of a $SU(2)$ group element.

Theorem 2. *The leading asymptotic behavior of the character of a $SU(2)$ group element (with Euler angles (α, β, γ)) in the J representation, $\chi^J(\alpha, \beta, \gamma)$ is*

$$\chi^J(\alpha, \beta, \gamma) \approx \frac{\sin \left[\left(J + \frac{1}{2} \right) \theta \right]}{\sin \frac{\theta}{2}}, \quad (34)$$

with θ defined by

$$\cos \frac{\theta}{2} = \cos \frac{\beta}{2} \cos \frac{(\alpha + \gamma)}{2}. \quad (35)$$

Let us emphasize that up to this point we already performed three different approximations: first the EM approximation, second the Stirling approximation and third the SPA approximation. To prove Theorem 2 we will use a second EM approximation and a second SPA approximation. However, formula (35) is exactly the classical relation between the Euler angle parameterization and the θ, \vec{n} parameterization of an $SU(2)$ group element, thus the leading behavior we find (after five levels of approximation) is in fact the exact formula of the character! We will discuss this rather surprising result in Section 6.

Proof of Theorem 2. To establish Theorem 2 we follow again the EM SPA recipe. The character χ^J of a group element writes

$$\chi^J(\alpha, \beta, \gamma) = \sum_{M=-J}^J D_{MM}^J(\alpha, \beta, \gamma) = \sum_{x=-1}^1 D_{xJ,xJ}^J(\alpha, \beta, \gamma), \quad (36)$$

with $x = \frac{M}{J}$ the rescaled variable. Note that the step in the second sum is $dx = \frac{1}{J}$. The leading EM approximation (see end of Appendix E) for the character is therefore the continuous integral (dropping henceforth the argument (α, β, γ))

$$\chi^J \approx J \int_{-1}^1 dx D_{xJ,xJ}^J. \quad (37)$$

We now use Theorem 1 (more precisely eq. (33)) and write a diagonal Wigner matrix element as

$$D_{xJ,xJ}^J \approx \left[\frac{1}{4\pi J\sqrt{\Delta}} \right]^{\frac{1}{2}} \left[\sqrt{\frac{e^{\imath\phi}}{\imath}} e^{Jf(x,x,u_+)} + \sqrt{\frac{e^{-\imath\phi}}{-\imath}} e^{Jf(x,x,u_-)} \right]. \quad (38)$$

Note that for diagonal matrix elements the exponents simplify to

$$f(x, x, u_{\pm}) = -\imath(\alpha + \gamma)x \pm \imath(\phi + x(\psi - \omega)), \quad (39)$$

while the discriminant Δ and angles ϕ , ψ and ω from eq. (11) become

$$\imath\phi = \ln \frac{2\xi^2 - 1 - x^2 + 2\imath\sqrt{\Delta}}{(1 - x^2)}, \quad \imath\psi = \ln \frac{x(1 - \xi^2) + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1 - \xi^2)}(1 - x^2)}, \quad (40)$$

$$\imath\omega = \ln \frac{-x(1 - \xi^2) + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1 - \xi^2)}(1 - x^2)}, \quad \Delta = (1 - \xi^2)(\xi^2 - x^2). \quad (41)$$

We follow the same steps as in the proof of Theorem 1.

Critical set \mathcal{C}_χ . The derivatives of the exponents for each of the two terms in eq. (38) are

$$\partial_x f(x, x, u_{\pm}) = -\imath(\alpha + \gamma) \pm \imath(\psi - \omega) \pm \imath\partial_x \phi \pm \imath x \partial_x(\psi - \omega). \quad (42)$$

The derivative of ϕ computes to

$$\imath\partial_x \phi = \partial_x \left[\ln(\sqrt{\xi^2 - x^2} + \imath\sqrt{1 - \xi^2})^2 - \ln(1 - x^2) \right] = \imath \frac{2x\sqrt{1 - \xi^2}}{(1 - x^2)\sqrt{\xi^2 - x^2}}. \quad (43)$$

The difference $\psi - \omega$ computes to

$$\imath(\psi - \omega) = \ln \frac{x(1 - \xi^2) + \imath\sqrt{\Delta}}{-x(1 - \xi^2) + \imath\sqrt{\Delta}} = \ln \frac{(\sqrt{\xi^2 - x^2} - \imath x\sqrt{1 - \xi^2})^2}{\xi^2(1 - x^2)}, \quad (44)$$

and its derivative

$$\imath\partial_x(\psi - \omega) = 2 \frac{\frac{-x}{\sqrt{\xi^2 - x^2}} - \imath\sqrt{1 - \xi^2}}{\sqrt{\xi^2 - x^2} - \imath x\sqrt{1 - \xi^2}} - \frac{-2x}{1 - x^2} = \imath \frac{-2\sqrt{1 - \xi^2}}{(1 - x^2)\sqrt{\xi^2 - x^2}}. \quad (45)$$

Combining eq. (43) and (45) we have

$$\partial_x \phi + x \partial_x(\psi - \omega) = 0, \quad (46)$$

and the saddle point equations (42) simplify to

$$\psi - \omega = \pm(\alpha + \gamma). \quad (47)$$

Dividing by 2 and exponentiating we get

$$\frac{\sqrt{\xi^2 - x^2} - \imath x\sqrt{1 - \xi^2}}{\sqrt{\xi^2(1 - x^2)}} = e^{\pm \imath \frac{\alpha + \gamma}{2}} \Rightarrow \frac{x\sqrt{1 - \xi^2}}{\sqrt{\xi^2 - x^2}} = \mp \tan \frac{\alpha + \gamma}{2}. \quad (48)$$

Hence the saddle points are solutions of the quadratic equation

$$x^2(1 - \xi^2) = (\xi^2 - x^2) \tan^2 \frac{\alpha + \gamma}{2} \Rightarrow x^2 = \frac{\xi^2 \sin^2 \frac{\alpha + \gamma}{2}}{1 - \xi^2 \cos^2 \frac{\alpha + \gamma}{2}}. \quad (49)$$

Defining a new variable θ via the relation $\cos \frac{\theta}{2} = \xi \cos \frac{\alpha + \gamma}{2}$ the saddle points rewrite

$$x^2 = \frac{\xi^2 \sin^2 \frac{\alpha + \gamma}{2}}{\sin^2 \frac{\theta}{2}}. \quad (50)$$

Taking into account eq. (48) one identifies an unique saddle point (x_1) for $f(x, x, u_+)$ and an unique saddle point (x_2) for $f(x, x, u_-)$ with x_1 and x_2 given by

$$x_1 = -\frac{\xi \sin \frac{\alpha + \gamma}{2}}{\sin \frac{\theta}{2}}, \quad x_2 = \frac{\xi \sin \frac{\alpha + \gamma}{2}}{\sin \frac{\theta}{2}}. \quad (51)$$

Evaluation of the functions and Hessian on \mathcal{C}_χ . Straightforward computations lead to

$$\xi^2 - x_{1,2}^2 = (1 - \xi^2) \frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}, \quad \Delta|_{x_{1,2}} = (1 - \xi^2)^2 \frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} \geq 0, \quad 1 - x_{1,2}^2 = \frac{(1 - \xi^2)}{\sin^2 \frac{\theta}{2}}. \quad (52)$$

Also note that at the saddle points ϕ simplifies as

$$\begin{aligned} \imath\phi &= \ln \frac{2\xi^2 - 1 - x_{1,2}^2 + 2\imath\sqrt{\Delta|_{x_{1,2}}}}{(1 - x_{1,2}^2)} = \ln \frac{(1 - \xi^2) \frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} - (1 - \xi^2) + 2\imath(1 - \xi^2) \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}}{\frac{(1 - \xi^2)}{\sin^2 \frac{\theta}{2}}} \\ &= \ln \left[\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} + \imath 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} \right] = \ln e^{i\theta} = i\theta. \end{aligned} \quad (53)$$

Substituting the saddle point equations (47) into eq. (39), we see that, at the saddles

$$f(x_1, x_1, u_+) = \imath\phi = i\theta, \quad f(x_2, x_2, u_-) = -\imath\phi = -i\theta. \quad (54)$$

To evaluate the Hessian at the saddle we first simplify eq. (42) using eq. (46) hence

$$\partial_x^2 f(x, x, u_\pm) = \pm \imath \partial_x (\psi - \omega) = \mp 2\imath \frac{\sqrt{1 - \xi^2}}{(1 - x^2) \sqrt{\xi^2 - x^2}} \quad (55)$$

which becomes at the saddle points

$$\mp 2\imath \frac{\sqrt{1 - \xi^2}}{\frac{(1 - \xi^2)}{\sin^2 \frac{\theta}{2}} \sqrt{(1 - \xi^2) \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}}} = \mp 2\imath \frac{1}{1 - \xi^2} \frac{\sin^3 \frac{\theta}{2}}{\cos \frac{\theta}{2}}. \quad (56)$$

Final evaluation. Using eq. (54) and eq. (56), the SPA of the character eq. (37) is

$$\chi^J \approx \frac{1}{\sqrt{2(1 - \xi^2) \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}}} \left(\sqrt{\frac{e^{i\theta}}{\imath}} \frac{e^{\imath J\theta}}{\sqrt{\imath \frac{2}{1 - \xi^2} \frac{\sin^3 \frac{\theta}{2}}{\cos \frac{\theta}{2}}}} + \sqrt{\frac{e^{-i\theta}}{-\imath}} \frac{e^{-\imath J\theta}}{\sqrt{-\imath \frac{2}{1 - \xi^2} \frac{\sin^3 \frac{\theta}{2}}{\cos \frac{\theta}{2}}}} \right), \quad (57)$$

which is

$$\chi^J \approx \frac{1}{2 \sin \frac{\theta}{2}} \left(\frac{1}{\imath} e^{\imath(J + \frac{1}{2})\theta} + \frac{1}{-\imath} e^{-\imath(J + \frac{1}{2})\theta} \right) = \frac{\sin \left[(J + \frac{1}{2})\theta \right]}{\sin \frac{\theta}{2}} \quad (58)$$

□

5 Asymptotes of $3j$ symbols

In this section we employ the asymptotic formula for the Wigner matrices to obtain an asymptotic formula for Wigner's $3j$ symbol. Note that one can use directly the EM SPA method to derive this asymptotic starting from the single sum representation of the $3j$ symbol [20]. We take here the alternative route of using the results of Theorem 1 and the representation of $3j$ symbols in terms of Wigner matrices

$$\int dg D_{M_1 M'_1}^{J_1}(g) D_{M_2 M'_2}^{J_2}(g) D_{M_3 M'_3}^{J_3}(g) = \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ M'_1 & M'_2 & M'_3 \end{pmatrix}, \quad (59)$$

where the integral is taken over $SU(2)$ with the normalized Haar measure

$$\int dg := \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^\pi d\beta \sin \beta. \quad (60)$$

Substituting the asymptote (9) for each matrix element $D_{M_i M'_i}^{J_i}(g)$ ($i = 1, 2, 3$), the integral (59) becomes

$$\int dg \left(\frac{1}{4\pi J_1 \sqrt{\Delta_1}} \right)^{1/2} \left(\frac{1}{4\pi J_2 \sqrt{\Delta_2}} \right)^{1/2} \left(\frac{1}{4\pi J_3 \sqrt{\Delta_3}} \right)^{1/2} \prod_{i=1}^3 \sum_{s_i=\pm 1} e^{-iJ_i(\alpha+\gamma)} \frac{1}{\sqrt{s_i l}} e^{i s_i \left(\frac{\phi_i}{2} + J_i (\phi_i + x_i \psi_i - y_i \omega_i) \right)}. \quad (61)$$

We expand (61), perform the integration over α and γ and change variables from β to ξ

$$\frac{1}{2} \int_0^\pi \sin \beta d\beta = \frac{1}{2} \int_0^\pi 2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} d\beta = 2 \int_0^1 \xi d\xi = \int_0^1 d(\xi^2), \quad (62)$$

to rewrite it as

$$\delta_{\sum_i J_i x_i, 0} \delta_{\sum_i J_i y_i, 0} \left[\int_0^1 d(\xi^2) \right] \left(\frac{1}{(4\pi)^3 \prod_i J_i \sqrt{\prod_i \Delta_i}} \right)^{\frac{1}{2}} \sum_{s_i=\pm 1} \frac{1}{\sqrt{\prod_i s_i l^3}} e^{i \sum_i s_i \left(\frac{\phi_i}{2} + f_i \right)}, \quad (63)$$

where the index i runs from 1 to 3, $\delta_{\sum_i J_i x_i, 0}$ is a Kronecker symbols and f_i is

$$f_i = J_i [\phi_i + x_i \psi_i - y_i \omega_i]. \quad (64)$$

We will derive the asymptotic behavior of eq. (63) via an SPA with respect to ξ^2 . Note that eq. (59) involves two distinct $3j$ symbols. If one attempts to first set $M'_i = M_i$, and obtain a representation of the square of a single $3j$ symbol, one encounters a very serious technical problem. We will see in the sequel that there are two saddle points ξ_\pm^2 contributing to the asymptotic behavior of eq. (63). If one starts by setting $M_i = M'_i$, one of the two saddle points $\xi_+^2 = 1$, and the second derivative in ξ_+^2 diverges. The contribution of this saddle point does not evaluate by a simple Gaussian integration.

The SPA evaluation of the general case, eq. (63), is a very lengthy computation. We will perform it using the classical angular momentum vectors. For large representation index J_i ,

there exists a classical angular momentum vector \vec{J}_i in \mathbb{R}^3 of length $|\vec{J}_i| = J_i$ and projection on the Oz axis (of unit vector \vec{n}) $\vec{n} \cdot \vec{J}_i = M_i$. A $3j$ symbol is then associated to three vectors, $\vec{J}_1, \vec{J}_2, \vec{J}_3$ with $|\vec{J}_i| = J_i$ and $\vec{n} \cdot \vec{J}_i = M_i = x_i J_i$. By the selection rules the quantum numbers J_i respect the triangle inequalities, and $M_1 + M_2 + M_3 = 0$. This translate into the condition that the vectors \vec{J}_i form a triangle $\vec{J}_1 + \vec{J}_2 + \vec{J}_3 = 0$ (and $\vec{n} \cdot [\vec{J}_1 + \vec{J}_2 + \vec{J}_3] = 0$). The asymptotic behavior of the $3j$ symbol writes in terms of the angular momentum vectors as

Theorem 3. *For large representation indices J_i the $3j$ symbol has the asymptotic behavior*

$$\begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} = \frac{1}{\sqrt{\pi(\vec{n} \cdot \vec{S})}} \cos \left[\sum_i (J_i + \frac{1}{2}) \Phi_{\vec{n}}^i + (\vec{n} \cdot \vec{J}_1) \Psi_{\vec{n}}^{13} + (\vec{n} \cdot \vec{J}_2) \Psi_{\vec{n}}^{23} + \frac{\pi}{4} \right], \quad (65)$$

with $\vec{S} = \vec{J}_1 \wedge \vec{J}_2 = \vec{J}_2 \wedge \vec{J}_3 = \vec{J}_3 \wedge \vec{J}_1$, twice the area of the triangle $\{\vec{J}_i\}$ and $\Phi_{\vec{n}}^i$ and $\Psi_{\vec{n}}^{13}$ and $\Psi_{\vec{n}}^{23}$ five angles defined as

$$\begin{aligned} \imath \Phi_{\vec{n}}^i &= \ln \frac{\vec{n} \cdot (\vec{J}_i \wedge \vec{S}) + \imath J_i (\vec{n} \cdot \vec{S})}{S \sqrt{(\vec{n} \wedge \vec{J}_i)^2}}, \\ \imath \Psi_{\vec{n}}^{i3} &= \ln \frac{(\vec{n} \wedge \vec{J}_i) \cdot (\vec{n} \wedge \vec{J}_3) + \imath \vec{n} \cdot (\vec{J}_3 \wedge \vec{J}_i)}{\sqrt{(\vec{n} \wedge \vec{J}_i)^2 (\vec{n} \wedge \vec{J}_3)^2}}, \quad i = 1, 2. \end{aligned} \quad (66)$$

Before proceeding with the proof of Theorem 3 note that our starting equation (59) involves two distinct $3j$ symbols. They are each associated to a triple of vectors, $\vec{J}_1, \vec{J}_2, \vec{J}_3$ ($|\vec{J}_i| = J_i$ and $\vec{n} \cdot \vec{J}_i = x_i J_i$) and $\vec{J}'_1, \vec{J}'_2, \vec{J}'_3$ ($|\vec{J}'_i| = J_i$, $\vec{n} \cdot \vec{J}'_i = y_i J_i$). Note that $|\vec{J}_i| = |\vec{J}'_i|$ hence the two triangles $\{\vec{J}_i\}$ and $\{\vec{J}'_i\}$ are congruent. Consequently there exists a rotation which overlaps them. Under this rotation the normal vector \vec{n} turns into the unit vector \vec{k} . All the geometrical information can therefore be encoded into an *unique* triple of vectors, henceforth denoted \vec{J}_i , and the *two* unit vectors \vec{n} and \vec{k} such that $|\vec{J}_i| = J_i$, $\vec{n} \cdot \vec{J}_i = x_i J_i$ and $\vec{k} \cdot \vec{J}_i = y_i J_i$ (see figure 1).

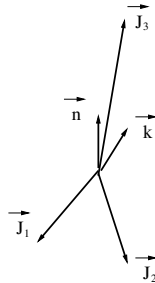


Figure 1: Angular momentum vectors

Proof of Theorem 3: The proof follows the by now familiar routine of an SPA. We perform this evaluation at fixed angular momenta, that is at fixed set of vectors $\vec{J}_i, \vec{n}, \vec{k}$.

The dominant saddle points: The saddle points governing the asymptotic behavior of eq. (63) are solutions of the equation

$$0 = \partial_{(\xi^2)} \sum s_i (\imath f_i) = \imath \sum_i s_i J_i [\partial_{(\xi^2)} \phi_i + x_i \partial_{(\xi^2)} \psi_i - y_i \partial_{(\xi^2)} \omega_i] . \quad (67)$$

A straightforward computation (see Appendix D.1) yields

$$\partial_{(\xi^2)} \sum s_i (\imath f_i) = -\frac{\imath}{\xi^2(1-\xi^2)} \sum_i s_i J_i \sqrt{\Delta_i} , \quad (68)$$

hence the saddle point equation writes

$$0 = s_1 J_1 \sqrt{\Delta_1} + s_2 J_2 \sqrt{\Delta_2} + s_3 J_3 \sqrt{\Delta_3} . \quad (69)$$

Introducing the angular momentum vectors the saddle point equation becomes after a short computation (see Appendix D.2)

$$4\xi^4 S^2 - 4\xi^2 \left\{ S^2 + (\vec{n} \cdot \vec{k}) S^2 - (\vec{n} \cdot \vec{S})(\vec{k} \cdot \vec{S}) \right\} + \left\{ [1 + (\vec{n} \cdot \vec{k})]^2 S^2 - 2(\vec{n} \cdot \vec{S})(\vec{k} \cdot \vec{S}) [1 + (\vec{n} \cdot \vec{k})] \right\} = 0 , \quad (70)$$

for all choices of signs s_1, s_2 and s_3 . Dividing by $4S^2$, eq. (70) factors as

$$\left[\xi^2 - \frac{1 + (\vec{n} \cdot \vec{k})}{2} \right] \left[\xi^2 - \left(\frac{1 + (\vec{n} \cdot \vec{k})}{2} - \frac{(\vec{n} \cdot \vec{S})(\vec{k} \cdot \vec{S})}{S^2} \right) \right] = 0 , \quad (71)$$

with roots

$$\xi_+^2 = \frac{1 + (\vec{n} \cdot \vec{k})}{2} , \quad \xi_-^2 = \frac{1 + (\vec{n} \cdot \vec{k})}{2} - \frac{(\vec{n} \cdot \vec{S})(\vec{k} \cdot \vec{S})}{S^2} , \quad (72)$$

again independent of the signs s_1, s_2 and s_3 . To identify the terms contributing to the asymptotic of eq. (63) for fixed \vec{J}_i, \vec{n} and \vec{k} one needs to evaluate $J_i \sqrt{\Delta_i}$ for each of the two roots ξ_+^2 and ξ_-^2 . Using Appendix D.3, we have

$$J_i^2 \Delta_i^+ = \frac{1}{4} \left[\vec{J}_i \cdot (\vec{n} \wedge \vec{k}) \right]^2 , \quad J_i^2 \Delta_i^- = \frac{1}{4} \frac{\left\{ \vec{J}_i \cdot \left[(\vec{S} \wedge \vec{n})(\vec{k} \cdot \vec{S}) + (\vec{S} \wedge \vec{k})(\vec{n} \cdot \vec{S}) \right] \right\}^2}{S^4} . \quad (73)$$

To any semiclassical state $\vec{J}_i, \vec{n}, \vec{k}$ we associate six signs, ϵ_i^+ and ϵ_i^- defined by

$$J_i \sqrt{\Delta_i^+} = \epsilon_i^+ \frac{1}{2} \vec{J}_i \cdot (\vec{n} \wedge \vec{k}) , \quad J_i \sqrt{\Delta_i^-} = \epsilon_i^- \frac{1}{2} \frac{\vec{J}_i \cdot \left[(\vec{S} \wedge \vec{n})(\vec{k} \cdot \vec{S}) + (\vec{S} \wedge \vec{k})(\vec{n} \cdot \vec{S}) \right]}{S^2} . \quad (74)$$

Substituting $J_i \sqrt{\Delta_i^\pm}$ into the saddle point eq. (69) the latter becomes

$$\frac{1}{2} \left(\sum_i s_i \epsilon_i^\pm \vec{J}_i \right) \cdot \vec{A}^\pm , \quad (75)$$

with $\vec{A}^+ = (\vec{n} \wedge \vec{k})$ and $\vec{A}^- = \frac{[(\vec{S} \wedge \vec{n})(\vec{k} \cdot \vec{S}) + (\vec{S} \wedge \vec{k})(\vec{n} \cdot \vec{S})]}{S^2}$. As, on the other hand, $\sum_i \vec{J}_i = 0$, we conclude that at fixed semiclassical state we have two saddle points ξ_+^2 and two saddle points ξ_-^2 contributing

- The ξ_+^2 saddle point in the term $s_i = \epsilon_i^+$ and in the term $s_i = -\epsilon_i^+$
- The ξ_-^2 saddle point in the term $s_i = \epsilon_i^-$ and in the term $s_i = -\epsilon_i^-$

The SPA evaluation of eq. (63) is the sum of this four contributions.

The second derivative: The derivative of eq. (68) with respect to ξ^2 yields

$$\begin{aligned} \partial_{(\xi^2)}[\partial_{(\xi^2)} \sum_i s_i(\imath f_i)] &= -\imath \partial_{(\xi^2)} \left(\frac{1}{\xi^2(1-\xi^2)} \right) \sum_i s_i J_i \sqrt{\Delta_i} \\ &\quad - \frac{\imath}{\xi^2(1-\xi^2)} \sum_i s_i J_i \frac{-(2\xi^2 - 1 - x_i y_i)}{2\sqrt{\Delta_i}}, \end{aligned} \quad (76)$$

and the term in the first line cancels (due to the saddle point equation) when evaluating the second derivative at the critical points. After Gaussian integration of the dominant saddle point contributions, the prefactor in the SPA approximation of eq. (63) writes

$$\frac{1}{\sqrt{K}}, \quad K = 32 \pi^2 s_1 s_2 s_3 \imath^3 J_1 J_2 J_3 \sqrt{\Delta_1 \Delta_2 \Delta_3} \left(-\partial_{(\xi^2)}^2 \sum_i s_i(\imath f_i) \right). \quad (77)$$

The reminder of this paragraphs is devoted to the evaluation of the K for the two roots ξ_+^2 and ξ_-^2 . Substituting the second derivative yields

$$K^\pm = -16\pi^2 s_1 s_2 s_3 \imath^4 \frac{J_1 J_2 J_3}{\xi_\pm^2(1-\xi_\pm^2)} \sum_i s_i J_i \frac{\sqrt{\Delta_1 \Delta_2 \Delta_3}}{\sqrt{\Delta_i}} (2\xi_\pm^2 - 1 - x_i y_i). \quad (78)$$

Taking into account $s_1^2 s_2 s_3 = \epsilon_2^\pm \epsilon_3^\pm$, K^\pm writes

$$K^\pm = -(16\pi^2) \frac{\left[\epsilon_2^\pm \epsilon_3^\pm J_2 \sqrt{\Delta_2^\pm} J_3 \sqrt{\Delta_3^\pm} [(2\xi_\pm^2 - 1)J_1^2 - J_1^\vec{n} J_1^\vec{k}] + \mathcal{O}_{123} \right]}{\xi_\pm^2(1-\xi_\pm^2)}, \quad (79)$$

where \mathcal{O}_{123} denotes circular permutations on the indices 1, 2 and 3. Using eq. (72), the denominator evaluates, for the ξ_+^2 root,

$$\xi_+^2(1-\xi_+^2) = \frac{1 - (\vec{n} \cdot \vec{k})^2}{4}, \quad (80)$$

while the numerator computes to (see Appendix D.4 for detailed computations and notations)

$$\epsilon_2^+ \epsilon_3^+ J_2 \sqrt{\Delta_2^+} J_3 \sqrt{\Delta_3^+} [(2\xi_+^2 - 1)J_1^2 - J_1^\vec{n} J_1^\vec{k}] + \mathcal{O}_{123} = -\frac{1}{4} S^\vec{n} S^\vec{k} (\vec{n} \wedge \vec{k})^2, \quad (81)$$

hence

$$K^+ = 16\pi^2 S^\vec{n} S^\vec{k}. \quad (82)$$

Evaluating the denominator in eq. (79) for ξ_-^2 yields

$$\xi_-^2(1-\xi_-^2) = \left(\frac{1 + (\vec{n} \cdot \vec{k})}{2} - \frac{S^\vec{n} S^\vec{k}}{S^2} \right) \left(\frac{1 - (\vec{n} \cdot \vec{k})}{2} + \frac{S^\vec{n} S^\vec{k}}{S^2} \right)$$

$$= \frac{1}{4} \left\{ (1 - (\vec{n} \cdot \vec{k})^2 + 4(\vec{n} \cdot \vec{k}) \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} - 4 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^4} \right\}, \quad (83)$$

while a lengthy computation (see Appendix D.4) shows that the numerator is

$$\begin{aligned} & \epsilon_2^- \epsilon_3^- J_2 \sqrt{\Delta_2^-} J_3 \sqrt{\Delta_3^-} \left[(2\xi_-^2 - 1) J_1^2 - J_1^{\vec{n}} J_1^{\vec{k}} \right] + \odot_{123} = \\ & = \frac{1}{4} S^{\vec{n}} S^{\vec{k}} \left\{ 1 - (\vec{n} \cdot \vec{k})^2 + 4(\vec{n} \cdot \vec{k}) \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} - 4 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^4} \right\}, \end{aligned} \quad (84)$$

proving that

$$K^- = -16\pi^2 S^{\vec{n}} S^{\vec{k}}. \quad (85)$$

Contribution of each saddle: To evaluate the contribution of each saddle point to the asymptote of eq. (63) we first evaluate

$$\imath \sum_i s_i \left[\frac{\phi_i}{2} + f_i \right] = \sum_i s_i \left[\left(J_i + \frac{1}{2} \right) (\imath \phi_i^\pm) + x_i J_i (\imath \psi_i^\pm) - y_i J_i (\imath \omega_i^\pm) \right]. \quad (86)$$

Recall that for a fixed semiclassical state only the terms with s_i equal to ϵ_i^+ , $-\epsilon_i^+$, ϵ_i^- and $-\epsilon_i^-$ contribute. We substitute $x_3 J_3 = -x_2 J_2 - x_1 J_1$ and $y_3 J_3 = -y_1 J_1 - y_2 J_2$ into eq. (86) to bring it into the form

$$\begin{aligned} \pm \quad & \left\{ \sum_i \left(J_i + \frac{1}{2} \right) (\imath \epsilon_i^\pm \phi_i^\pm) + x_1 J_1 (\imath \epsilon_1^\pm \psi_1^\pm - \imath \epsilon_3^\pm \psi_3^\pm) + x_2 J_2 (\imath \epsilon_2^\pm \psi_2^\pm - \imath \epsilon_3^\pm \psi_3^\pm) \right. \\ & \left. - y_1 J_1 (\imath \epsilon_1^\pm \omega_1^\pm - \imath \epsilon_3^\pm \omega_3^\pm) - y_2 J_2 (\imath \epsilon_2^\pm \omega_2^\pm - \imath \omega_3^\pm \psi_3^\pm) \right\}, \end{aligned} \quad (87)$$

where ϕ_i^\pm , ψ_i^\pm and ω_i^\pm are the angles ϕ_i , ψ_i and ω_i evaluated at ξ_+^2 and ξ_-^2 . For each choice $+$ or $-$ in the accolades, one must count *both* choices of the overall sign. The angles ϕ_i^\pm , $\epsilon_1^\pm \psi_1^\pm - \epsilon_3^\pm \psi_3^\pm$, etc. are evaluated by a rather involved computation in Appendix D.5. The end results are synthesized below

$$\begin{aligned} \imath \epsilon_i^\pm \phi_i^\pm &= \imath \Phi_{\vec{n}}^i \mp \imath \Phi_{\vec{k}}^i, \quad \imath \Phi_{\vec{n}}^i = \ln \frac{\vec{n} \cdot (\vec{J}_i \wedge \vec{S}) + \imath J_i S^{\vec{n}}}{S \sqrt{(\vec{n} \wedge \vec{J}_i)^2}} \\ \imath \epsilon_j^\pm \psi_j^\pm - \imath \epsilon_3^\pm \psi_3^\pm &= \imath \Psi_{\vec{n}}^{j3}, \quad \imath \Psi_{\vec{n}}^{j3} = \ln \frac{(\vec{n} \wedge \vec{J}_j) \cdot (\vec{n} \wedge \vec{J}_3) + \imath \vec{n} \cdot (\vec{J}_3 \wedge \vec{J}_j)}{\sqrt{(\vec{n} \wedge \vec{J}_j)^2 (\vec{n} \wedge \vec{J}_3)^2}}, \quad j = 1, 2, \\ \imath \epsilon_j^\pm \omega_j^\pm - \imath \epsilon_3^\pm \omega_3^\pm &= \pm \imath \Psi_{\vec{k}}^{j3}. \end{aligned} \quad (88)$$

Substituting eq. (88) into eq. (87) yields

$$\begin{aligned} \pm \quad & \left\{ \sum_i \left(J_i + \frac{1}{2} \right) (\imath \Phi_{\vec{n}}^i \mp \imath \Phi_{\vec{k}}^i) + (\vec{n} \cdot \vec{J}_1) \imath \Psi_{\vec{n}}^{13} + (\vec{n} \cdot \vec{J}_2) \imath \Psi_{\vec{n}}^{23} \right. \\ & \left. \mp (\vec{k} \cdot \vec{J}_1) \imath \Psi_{\vec{k}}^{13} \mp (\vec{k} \cdot \vec{J}_2) \imath \Psi_{\vec{k}}^{23} \right\} = \pm (\Omega_{\vec{n}} \mp \Omega_{\vec{k}}), \end{aligned} \quad (89)$$

where $\Omega_{\vec{n}}$ denotes

$$\imath\Omega_{\vec{n}} = \sum_i \left(J_i + \frac{1}{2}\right) \imath\Phi_{\vec{n}}^i + (\vec{n} \cdot \vec{J}_1) \imath\Psi_{\vec{n}}^{13} + (\vec{n} \cdot \vec{J}_2) \imath\Psi_{\vec{n}}^{23} . \quad (90)$$

Final evaluation: We put together eq. (82), (85) and (89) and, noting that the two contributions from the saddle ξ_{\pm}^2 are complex conjugate to one another we obtain

$$\begin{aligned} & \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ M'_1 & M'_2 & M'_3 \end{pmatrix} \\ & \approx \frac{1}{\sqrt{\pi(\vec{n} \cdot \vec{S})}} \frac{1}{\sqrt{\pi(\vec{k} \cdot \vec{S})}} \frac{1}{4} \left(e^{\imath(\Omega_{\vec{n}} - \Omega_{\vec{k}})} + e^{-\imath(\Omega_{\vec{n}} - \Omega_{\vec{k}})} + \imath e^{\imath(\Omega_{\vec{n}} + \Omega_{\vec{k}})} - \imath e^{-\imath(\Omega_{\vec{n}} + \Omega_{\vec{k}})} \right) . \end{aligned} \quad (91)$$

Taking into account

$$\frac{1}{4} \left(e^{\imath(\Omega_{\vec{n}} - \Omega_{\vec{k}})} + e^{-\imath(\Omega_{\vec{n}} - \Omega_{\vec{k}})} + \imath e^{\imath(\Omega_{\vec{n}} + \Omega_{\vec{k}})} - \imath e^{-\imath(\Omega_{\vec{n}} + \Omega_{\vec{k}})} \right) = \cos\left(\Omega_{\vec{n}} + \frac{\pi}{4}\right) \cos\left(\Omega_{\vec{k}} + \frac{\pi}{4}\right) , \quad (92)$$

Theorem 3 follows. □

6 Conclusion

Using the EM SPA method we have determined the asymptotic behaviors at large spin J of Wigner matrix elements, Wigner $3j$ symbols and the character $\chi^J(g)$ of an $SU(2)$ group element g .

By far the most surprising fact about this computation is that our formula for the character $\chi^J(g)$ is exact. SPA reproducing the exact result for integrals are usually the consequence of a Duistermaat Heckman [22, 23] localization property (one of the most famous example of this being the Harish Chandra Itzykson Zuber integral [24]). Recall that the Duistermaat-Heckman theorem states that a phase space integral

$$\int \Omega e^{-\imath H(p,q)} , \quad (93)$$

where Ω is the Liouville form, equals its leading order SPA estimation if the flow of the Hamiltonian vector field \mathbf{X} ($i_{\mathbf{X}}\Omega = dH$) is $U(1)$. To our knowledge all integrals exhibiting a localization property (i.e. equaling their leading order SPA approximation) fall in (some generalization of) this case. Note that the character of an $SU(2)$ group element can be expressed directly as a double integral by

$$\chi^J(g) = \sum_{M,t} e^{h(J,M,t)} \approx \frac{J}{2\pi} \int du dx \sqrt{K(x,x,u)} e^{Jf(x,x,u)} + \text{E.M.} + \text{S.} , \quad (94)$$

where E.M. denotes corrections coming from the Euler Maclaurin approximation, and S the corrections coming from sub leading terms in the Stirling approximation. The double integral

in equation (94) is of the correct form, with symplectic form $\Omega = \sqrt{K(x, x, u)} dx \wedge du$ and Hamiltonian $f(x, x, u)$ generating the Hamiltonian flow

$$\frac{du}{d\rho} = \sqrt{\frac{u^2(1+x-u)(1-x-u)}{1-x^2}} \ln \left\{ e^{-i(\alpha+\gamma)} \frac{(1+x)(1-x-u)}{(1-x)(1+x-u)} \right\} \quad (95)$$

$$\frac{dx}{d\rho} = -\sqrt{\frac{u^2(1+x-u)(1-x-u)}{1-x^2}} \ln \left\{ e^{i\pi} \frac{(1-\xi^2)}{\xi^2} \frac{(1-x-u)(1+x-u)}{u^2} \right\} . \quad (96)$$

Our result can be explained if first, the above flow is $U(1)$ (thus the SPA of the double integral is exact) and second the EM and Stirling correction terms cancel, E.M. + S. = 0. The alternative, namely that the flow is not $U(1)$ would require an even more subtle cancellation of the sub leading correction terms. Either way, the exact result for the character we derive in this paper deserves further investigation.

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Appendix

In these appendices we detail various technical points and computations.

A The Stirling approximation

We detail here the passage from eq. (17) to eq. (12). Our starting point is

$$D_{MM'}^J(\alpha, \beta, \gamma) \approx \int dt F(J, M, M', t) , \quad (A.1)$$

with

$$F(J, M, M', t) = e^{i\pi t} e^{-i\alpha M} e^{-i\gamma M'} \xi^{2J+M-M'-2t} \eta^{2t-M+M'} \times \frac{\sqrt{\Gamma(J+M+1)\Gamma(J-M+1)\Gamma(J+M'+1)\Gamma(J-M'+1)}}{\Gamma(J+M-t+1)\Gamma(J-M'-t+1)\Gamma(t+1)\Gamma(t-M+M'+1)} . \quad (A.2)$$

We use the Stirling formula

$$\Gamma(n+1) = n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n = \sqrt{2\pi n} e^{n \ln n - n} , \quad (A.3)$$

for all Γ functions and rescaled variables $M = xJ$, $M' = yJ$, $t = uJ$. Collecting all prefactors yields

$$\left(\frac{\sqrt{(2\pi)^4 J^4 (1+x)(1-x)(1+y)(1-y)}}{(2\pi)^4 J^4 (1+x-u)(1-y-u)(u)(u-x+y)} \right)^{\frac{1}{2}} = \frac{1}{2\pi J} \sqrt{K(x, y, u)} , \quad (A.4)$$

and $K(x, y, u)$ takes the form in eq. (14). The “-n” terms in the Stirling approximation add to

$$\begin{aligned} & \frac{1}{2} \left\{ -J(1+x) - J(1-x) - J(1+y) - J(1-y) \right\} \\ & - \left\{ -J(1+x-u) - J(1-y-u) - Ju - J(u-x+y) \right\} = 0, \end{aligned} \quad (\text{A.5})$$

which also implies that the coefficient of $\ln J$ in the exponent cancels. The contribution of the Γ functions eq. (A.2) is therefore

$$\begin{aligned} & \frac{J}{2} \left\{ (1+x) \ln(1+x) + (1-x) \ln(1-x) + (1+y) \ln(1+y) + (1-y) \ln(1-y) \right\} \\ & - J \left\{ (1+x-u) \ln(1+x-u) + (1-y-u) \ln(1-y-u) \right. \\ & \left. + u \ln(u) + (u-x+y) \ln(u-x+y) \right\}. \end{aligned} \quad (\text{A.6})$$

The substitution of eq. (A.6) into eq. (A.2) yields

$$F(J, xJ, yJ, uJ) \approx \frac{1}{2\pi J} \sqrt{K(x, y, u)} e^{Jf(x, y, u)}, \quad (\text{A.7})$$

where $f(x, y, u)$ takes the form in eq. (13), and

$$D_{MM'}^J(\alpha, \beta, \gamma) \approx \int dt F(J, M, M', t) \approx \int dt \frac{1}{2\pi J} \sqrt{K(x, y, u)} e^{Jf(x, y, u)}, \quad (\text{A.8})$$

which reproduces eq. (12) after changing the integration variable to $u = \frac{t}{J}$.

B Evaluations on the critical set.

In this appendix we present the various evaluations relevant for the proof of Theorem 1. We start by some preliminary computations. Recall that

$$\Delta = (1 - \xi^2)(\xi^2 - xy) - \frac{(x - y)^2}{4} \geq 0. \quad (\text{B.9})$$

As a preliminary we compute the absolute values of the four complex numbers

$$\begin{aligned} u_{\pm} &= 1 - \xi^2 + \frac{x - y}{2} \pm i\sqrt{\Delta}, & u_{\pm} - x + y &= 1 - \xi^2 - \frac{x - y}{2} \pm i\sqrt{\Delta}, \\ 1 + x - u_{\pm} &= \xi^2 + \frac{x + y}{2} \mp i\sqrt{\Delta}, & 1 - y - u_{\pm} &= \xi^2 - \frac{x + y}{2} \mp i\sqrt{\Delta}, \end{aligned} \quad (\text{B.10})$$

which are

$$\begin{aligned} |u_{\pm}|^2 &= (1 - \xi^2)(1 + x)(1 - y), & |u_{\pm} - x + y|^2 &= (1 - \xi^2)(1 - x)(1 + y), \\ |1 + x - u_{\pm}|^2 &= \xi^2(1 + x)(1 + y), & |1 - y - u_{\pm}|^2 &= \xi^2(1 - x)(1 - y). \end{aligned} \quad (\text{B.11})$$

B.1 Evaluation of f at the critical points

To establish eq. (25) and (26), we evaluate eq. (24) at u_{\pm}

$$\begin{aligned} f(x, y, u_{\pm}) = & -i\alpha x - i\gamma y + (2+x-y)\ln\xi + (-x+y)\ln\eta \\ & + \frac{1}{2}(1-x)\ln(1-x) + \frac{1}{2}(1+x)\ln(1+x) + \frac{1}{2}(1-y)\ln(1-y) + \frac{1}{2}(1+y)\ln(1+y) \\ & - (1+x)\ln(1+x-u_{\pm}) - (1-y)\ln(1-y-u_{\pm}) - (-x+y)\ln(u_{\pm}-x+y) . \end{aligned} \quad (\text{B.12})$$

The real part of $f(x, y, u_{\pm})$ is

$$\begin{aligned} \Re f(x, y, u_{\pm}) = & (2+x-y)\ln\xi + (-x+y)\ln\eta \\ & + \frac{1}{2}(1-x)\ln(1-x) + \frac{1}{2}(1+x)\ln(1+x) + \frac{1}{2}(1-y)\ln(1-y) + \frac{1}{2}(1+y)\ln(1+y) \\ & - (1+x)\ln|1+x-u_{\pm}| - (1-y)\ln|1-y-u_{\pm}| - (-x+y)\ln|u_{\pm}-x+y| , \end{aligned} \quad (\text{B.13})$$

and substituting the absolute values computed in eq. (B.11) yields

$$\begin{aligned} \Re f(x, y, u_{\pm}) = & (2+x-y)\ln\xi + (-x+y)\ln\eta \\ & + \frac{1}{2}(1-x)\ln(1-x) + \frac{1}{2}(1+x)\ln(1+x) + \frac{1}{2}(1-y)\ln(1-y) + \frac{1}{2}(1+y)\ln(1+y) \\ & - \frac{(1+x)}{2}\ln\left[\xi^2(1+x)(1+y)\right] - \frac{(1-y)}{2}\ln\left[\xi^2(1-x)(1-y)\right] \\ & - \frac{(-x+y)}{2}\ln\left[(1-\xi^2)(1-x)(1+y)\right] . \end{aligned} \quad (\text{B.14})$$

Recalling that $1-\xi^2=\eta^2$ we note that the coefficients of both $\ln\xi$ and $\ln(1-\xi^2)$ cancel. Furthermore, a direct inspection shows that the coefficients of all $\ln(1-x)$, $\ln(1+x)$, $\ln(1-y)$ and $\ln(1+y)$ cancel. Hence

$$\Re f(x, y, u_{\pm}) = 0 . \quad (\text{B.15})$$

Therefore $f(x, y, u_{\pm})$ is a purely imaginary number

$$\begin{aligned} f(x, y, u_{\pm}) = & -i\alpha x - i\gamma y - (1+x)\ln\frac{1+x-u_{\pm}}{|1+x-u_{\pm}|} - (1-y)\ln\frac{1-y-u_{\pm}}{|1-y-u_{\pm}|} \\ & - (-x+y)\ln\frac{u_{\pm}-x+y}{|u_{\pm}-x+y|} . \end{aligned} \quad (\text{B.16})$$

which assumes the form

$$f(x, y, u_{\pm}) = -i\alpha x - i\gamma y \pm i\left(\phi + x\psi - y\omega\right) , \quad (\text{B.17})$$

where the three angles ϕ , ψ and ω read

$$\begin{aligned} i\phi = & -\ln\frac{(1+x-u_+)}{|1+x-u_+|} - \ln\frac{(1-y-u_+)}{|1-y-u_+|} , \\ i\psi = & -\ln\frac{1+x-u_+}{|1+x-u_+|} + \ln\frac{(u_+-x+y)}{|u_+-x+y|} , \\ i\omega = & -\ln\frac{(1-y-u_+)}{|1-y-u_+|} + \ln\frac{u_+-x+y}{|u_+-x+y|} . \end{aligned} \quad (\text{B.18})$$

As the two roots u_+ and u_- are complex conjugate, one can absorb the various signs in eq. (B.18) and write

$$\begin{aligned} \imath\phi &= \ln \frac{(1+x-u_-)(1-y-u_-)}{|1+x-u_-||1-y-u_-|}, \quad \imath\psi = \ln \frac{(1+x-u_-)(u_+-x+y)}{|1+x-u_-||u_+-x+y|}, \\ \imath\omega &= \ln \frac{(1-y-u_-)(u_+-x+y)}{|1-y-u_-||u_+-x+y|}. \end{aligned} \quad (\text{B.19})$$

One by one ϕ , ψ and ω compute by substituting eq. (B.10) and eq. (B.11) to

$$\begin{aligned} \imath\phi &= \ln \frac{(\xi^2 + \frac{x+y}{2} + \imath\sqrt{\Delta})(\xi^2 - \frac{x+y}{2} + \imath\sqrt{\Delta})}{\sqrt{\xi^4(1-x^2)(1-y^2)}} \\ &= \ln \frac{\xi^4 - \frac{(x+y)^2}{4} + 2\xi^2\imath\sqrt{\Delta} - (1-\xi^2)(\xi^2 - xy) + \frac{(x-y)^2}{4}}{\sqrt{\xi^4(1-x^2)(1-y^2)}} \\ &= \ln \frac{2\xi^2 - 1 - xy + 2\imath\sqrt{\Delta}}{\sqrt{(1-x^2)(1-y^2)}}, \end{aligned} \quad (\text{B.20})$$

and

$$\begin{aligned} \imath\psi &= \ln \frac{(\xi^2 + \frac{x+y}{2} + \imath\sqrt{\Delta})(1-\xi^2 - \frac{x-y}{2} + \imath\sqrt{\Delta})}{\sqrt{\xi^2(1-\xi^2)(1-x^2)(1+y)^2}} \\ &= \ln \frac{\xi^2(1-\xi^2) + \frac{x+y}{2} - x\xi^2 - \frac{x^2-y^2}{4} - (1-\xi^2)(\xi^2 - xy) + \frac{(x-y)^2}{4} + \imath(1+y)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)(1+y)^2}} \\ &= \ln \frac{-x(1+y)\xi^2 + \frac{x+y}{2} + xy + \frac{y^2-xy}{2} + \imath(1+y)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)(1+y)^2}} \\ &= \ln \frac{\frac{x+y}{2} - x\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}}, \end{aligned} \quad (\text{B.21})$$

and finally

$$\begin{aligned} \imath\omega &= \ln \frac{(\xi^2 - \frac{x+y}{2} + \imath\sqrt{\Delta})(1-\xi^2 - \frac{x-y}{2} + \imath\sqrt{\Delta})}{\sqrt{\xi^2(1-\xi^2)(1-y^2)(1-x)^2}} \\ &= \ln \frac{\xi^2(1-\xi^2) - \frac{x+y}{2} + y\xi^2 + \frac{x^2-y^2}{4} - (1-\xi^2)(\xi^2 - xy) + \frac{(x-y)^2}{4} + \imath(1-x)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)(1-x)^2}} \\ &= \ln \frac{y(1-x)\xi^2 - \frac{x+y}{2} + xy + \frac{x^2-xy}{2} + \imath(1-x)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)(1-x)^2}} \\ &= \ln \frac{-\frac{x+y}{2} + y\xi + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)}}. \end{aligned} \quad (\text{B.22})$$

B.2 Evaluation of the second derivative.

From eq. (27) we have

$$-\partial_u^2 f(x, y, u) = \frac{1}{1+x-u} + \frac{1}{1-y-u} + \frac{1}{u} + \frac{1}{u-x+y}. \quad (\text{B.23})$$

Each term evaluates at the critical points as

$$\begin{aligned}
\frac{1}{1+x-u_{\pm}} &= \frac{1+x-u_{\mp}}{|1+x-u_{\pm}|^2} = \frac{\xi^2 + \frac{x+y}{2} \pm i\sqrt{\Delta}}{\xi^2(1+x)(1+y)} \\
\frac{1}{1-y+u_{\pm}} &= \frac{1-y+u_{\mp}}{|1-y+u_{\pm}|^2} = \frac{\xi^2 - \frac{x+y}{2} \pm i\sqrt{\Delta}}{\xi^2(1-x)(1-y)} \\
\frac{1}{u_{\pm}-x+y} &= \frac{u_{\mp}-x+y}{|u_{\mp}-x+y|^2} = \frac{(1-\xi^2) - \frac{x-y}{2} \mp i\sqrt{\Delta}}{(1-\xi^2)(1-x)(1+y)} \\
\frac{1}{u_{\pm}} &= \frac{u_{\mp}}{|u_{\mp}|^2} = \frac{(1-\xi^2) + \frac{x-y}{2} \mp i\sqrt{\Delta}}{(1-\xi^2)(1+x)(1-y)} .
\end{aligned} \tag{B.24}$$

The real part of (B.23) is therefore

$$\begin{aligned}
&\frac{\xi^2 + \frac{x+y}{2}}{\xi^2(1+x)(1+y)} + \frac{\xi^2 - \frac{x+y}{2}}{\xi^2(1-x)(1-y)} + \\
&\frac{(1-\xi^2) - \frac{x-y}{2}}{(1-\xi^2)(1-x)(1+y)} + \frac{(1-\xi^2) + \frac{x-y}{2}}{(1-\xi^2)(1+x)(1-y)} ,
\end{aligned} \tag{B.25}$$

and computes further to

$$\Re(-\partial_u^2 f)|_{x,y,u_{\pm}} = \frac{4\Delta}{(1-x^2)(1-y^2)\xi^2(1-\xi^2)} . \tag{B.26}$$

The imaginary part of eq. (B.23) is

$$\begin{aligned}
&\pm i\sqrt{\Delta} \left(\frac{1}{\xi^2(1+x)(1+y)} + \frac{1}{\xi^2(1-x)(1-y)} \right. \\
&\quad \left. - \frac{1}{(1-\xi^2)(1-x)(1+y)} - \frac{1}{(1-\xi^2)(1+x)(1-y)} \right) ,
\end{aligned} \tag{B.27}$$

which finally computes to

$$\Im(-\partial_u^2 f)|_{x,y,u_{\pm}} = \pm i2\sqrt{\Delta} \frac{1-2\xi^2-xy}{(1-x^2)(1-y^2)\xi^2(1-\xi^2)} . \tag{B.28}$$

B.3 Evaluation of K

The prefactor $K|_{x,y,u_{\pm}}$ is

$$K = \frac{\sqrt{(1-x^2)(1-y^2)}}{(1+x-u_{\pm})(1-y-u_{\pm})(u_{\pm})(u_{\pm}-x+y)} , \tag{B.29}$$

which is, using eq. (B.24),

$$\begin{aligned}
K = &\frac{\sqrt{(1-x^2)(1-y^2)}}{\xi^4(1-\xi^2)^2(1-x^2)^2(1-y^2)^2} \\
&\left(\xi^2 + \frac{x+y}{2} \pm i\sqrt{\Delta} \right) \left(\xi^2 - \frac{x+y}{2} \pm i\sqrt{\Delta} \right) \\
&\left((1-\xi^2) - \frac{x-y}{2} \mp i\sqrt{\Delta} \right) \left((1-\xi^2) + \frac{x-y}{2} \mp i\sqrt{\Delta} \right) ,
\end{aligned} \tag{B.30}$$

and a straightforward computation proves eq. (30).

C Real saddle points

In this section we present the SPA evaluation of a matrix element with

$$\Delta = (1 - \xi^2)(\xi^2 - xy) - \frac{(x - y)^2}{4} < 0. \quad (\text{C.31})$$

For convenience we denote $\Delta' = -\Delta > 0$. In this range of parameters the two saddle points

$$u_{\pm} = h_{\pm}(x; y) = (1 - \xi^2) + \frac{x - y}{2} \pm \sqrt{\Delta'}, \quad (\text{C.32})$$

are real. For simplicity suppose that $0 < x \leq y < 1$. A straightforward computation shows that $0 < u_- < u_+ < 1 - y$, hence both roots are in the integration interval. Using the results of Appendix B.1, the function evaluates at the two saddle points as

$$f|_{u_{\pm}} = -i\alpha x - i\beta y \pm (\Phi + x\Psi - y\Omega), \quad (\text{C.33})$$

with

$$\Phi = \ln \frac{(2\xi^2 - 1 - xy + 2\sqrt{\Delta'})}{\sqrt{(1 - x^2)(1 - y^2)}}, \quad (\text{C.34})$$

$$\Psi = \ln \frac{(-x\xi^2 + \frac{x+y}{2} + \sqrt{\Delta'})}{\sqrt{\xi^2(1 - \xi^2)(1 - x^2)}}, \quad (\text{C.35})$$

$$\Omega = \ln \frac{(\xi^2 y - \frac{x+y}{2} + \sqrt{\Delta'})}{\sqrt{\xi^2(1 - \xi^2)(1 - y^2)}}. \quad (\text{C.36})$$

From Appendix B.2, we obtain

$$-\partial_u^2 f|_{u_{\pm}} = \frac{-4\Delta' \mp 2\sqrt{\Delta'}(2\xi^2 - 1 - xy)}{\xi^2(1 - \xi^2)(1 - x^2)(1 - y^2)}, \quad (\text{C.37})$$

which shows in particular that the maximum of f is u_- (as $-\partial_u^2 f|_{u_-} < 0$), and the SPA is dominated by the latter. In Figure (2) below we represent the function $\Xi = \Phi + x\Psi - y\Omega$ as a function of x and y ,

The prefactor writes, using Appendix B.3,

$$K|_{u_-} = \frac{-\sqrt{(1 - x^2)(1 - y^2)}(2\xi^2 - 1 - xy - 2\sqrt{\Delta'})^2}{\xi^2(1 - \xi^2)(1 - x^2)^2(1 - y^2)^2}, \quad (\text{C.38})$$

hence we get the asymptotic estimate

$$D_{xJ,yJ}^J(\alpha, \beta, \gamma) \approx -\frac{1}{\sqrt{2\pi J}} \left(\frac{1}{2\sqrt{\Delta'}} \right)^{1/2} e^{-i\alpha Jx - i\gamma Jy} e^{-\frac{\Phi}{2}} e^{-J(\Phi + x\Psi - y\Omega)}, \quad (\text{C.39})$$

which indeed is suppressed for large J .

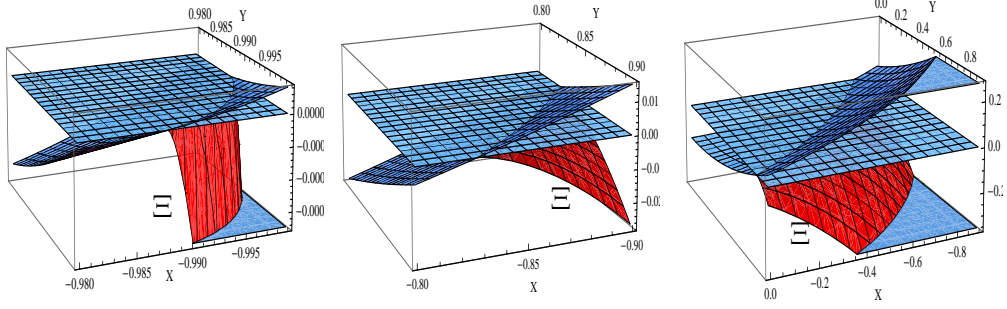


Figure 2: The function $\Xi = \Phi + x\Psi - y\Omega$ (red) is negative, and vanishes (plane $z = 0$, light blue) when Δ' (dark blue) vanishes, for $\xi = 0.1, 0.5$ and 0.9 , from the left to the right.

The case $\Delta' = 0$ is special. A straightforward calculation shows that under this circumstances

$$\Phi = \Psi = \Omega = 0 . \quad (\text{C.40})$$

Also, eq. (C.37) implies $\partial_u^2 f|_{u_{\pm}} = 0$. One needs to push the Taylor development around the root

$$u_0 = 1 - \xi^2 + \frac{x - y}{2} , \quad (\text{C.41})$$

to third order

$$f(u, x, y) = f|_{u_0} + \frac{1}{6}(u - u_0)^3 [\partial_u^3 f]|_{u_0} + O(u^3) , \quad (\text{C.42})$$

and the Wigner matrix elements has an asymptotic behavior (see [1])

$$\int du \sqrt{K(u, x, y)} e^{Jf} \approx e^{Jf|_{u_0}} \left\{ \text{Ai}(a(x, y)[iJ]^{\frac{2}{3}})[iJ]^{-\frac{1}{3}} + \text{Ai}'(a(x, y)[iJ]^{\frac{2}{3}})[iJ]^{-\frac{2}{3}} \right\} , \quad (\text{C.43})$$

where $a(x, y)$ is some non vanishing smooth real function (determined by K and f evaluated at u_0 , see [1]), Ai is the Airy function of the first kind and Ai' its derivative. At large argument the Airy functions behave like

$$\text{Ai}(\zeta) \approx \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}} \approx -\text{Ai}'(\zeta) . \quad (\text{C.44})$$

The term Ai' is therefore subleading and we have

$$\int du \sqrt{K(u, x, y)} e^{Jf} \approx \frac{e^{iJ\left(\alpha x + \gamma y - \frac{2}{3}(a(x, y))^{\frac{3}{2}}\right)}}{\sqrt{iJ}(a(x, y))^{1/4}} . \quad (\text{C.45})$$

D Computations for the $3j$ symbol

In this appendix we detail at length the various computations required for the proof of Theorem 3.

D.1 The first derivative

To compute the derivative $\partial_{\xi^2} \sum_i s_i(\imath f_i)$, note that $\partial_{(\xi^2)} \Delta_i = -(2\xi^2 - 1 - x_i y_i)$. The partial derivative of $\imath \phi_i$ is then

$$\begin{aligned} \imath \partial_{(\xi^2)} \phi_i &= \frac{2 + 2\imath \frac{\partial_{(\xi^2)} \Delta_i}{2\sqrt{\Delta_i}}}{(2\xi^2 - 1 - x_i y_i + 2\imath \sqrt{\Delta_i})} = \frac{(2 - \imath \frac{2\xi^2 - 1 - x_i y_i}{\sqrt{\Delta_i}})(2\xi^2 - 1 - x_i y_i - 2\imath \sqrt{\Delta_i})}{(2\xi^2 - 1 - x_i y_i)^2 + 4\Delta_i} \\ &= \frac{-\imath}{\sqrt{\Delta_i}}, \end{aligned} \quad (\text{D.46})$$

while the derivative of $\imath \psi_i$ writes

$$\begin{aligned} \imath \partial_{(\xi^2)} \psi_i &= \frac{-x_i + \imath \frac{\partial_{(\xi^2)} \Delta_i}{2\sqrt{\Delta_i}}}{\frac{x_i + y_i}{2} - x_i \xi^2 + \imath \sqrt{\Delta_i}} - \frac{1 - 2\xi^2}{2\xi^2(1 - \xi^2)} \\ &= \imath \frac{\left[-(2\xi^2 - 1 - x_i y_i) + \imath 2x_i \sqrt{\Delta_i} \right] \left[\frac{x_i + y_i}{2} - x_i \xi^2 - \imath \sqrt{\Delta_i} \right]}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2) (1 - x_i^2)} - \frac{1 - 2\xi^2}{2\xi^2(1 - \xi^2)}. \end{aligned} \quad (\text{D.47})$$

We first evaluate $2x_i \Delta_i - (2\xi^2 - 1 - x_i y_i) \left(\frac{x_i + y_i}{2} - x_i \xi^2 \right)$ as

$$\begin{aligned} &= 2x_i \left[-\xi^4 + \xi^2(1 + x_i y_i) - \frac{(x_i + y_i)^2}{4} \right] - [2\xi^2 - 1 - x_i y_i] \left(\frac{x_i + y_i}{2} - x_i \xi^2 \right) \\ &= \xi^2 \left[x_i(1 + x_i y_i) - x_i - y_i \right] - \frac{x_i + y_i}{2} [x_i(x_i + y_i) - 1 - x_i y_i] \\ &= (1 - x_i^2) \left(\frac{x_i + y_i}{2} - \xi^2 y_i \right), \end{aligned} \quad (\text{D.48})$$

hence eq. (D.47) writes

$$\begin{aligned} &\imath \frac{(1 - x_i^2) \left(\frac{x_i + y_i}{2} - \xi^2 y_i \right)}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2) (1 - x_i^2)} + \imath \frac{\imath \sqrt{\Delta_i} \left[x_i^2 - 2x_i^2 \xi^2 + 2\xi^2 - 1 \right]}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2) (1 - x_i^2)} - \frac{(1 - 2\xi^2)}{2\xi^2(1 - \xi^2)} \\ &= \imath \frac{\left(\frac{x_i + y_i}{2} - \xi^2 y_i \right)}{2\sqrt{\Delta_i} \xi^2}. \end{aligned} \quad (\text{D.49})$$

Noting that $\omega(x_i, y_i) = \psi(-y_i, -x_i)$ the derivative of $\imath \omega$ writes

$$\imath \partial_{(\xi^2)} \omega_i = \frac{\imath \left(-\frac{x_i + y_i}{2} + \xi^2 x_i \right)}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2)}. \quad (\text{D.50})$$

The derivative of $\sum_i s_i(\imath f_i)$ is then

$$\begin{aligned} \partial_{(\xi^2)} \sum_i s_i(\imath f_i) &= \imath \sum_i s_i J_i \left(\frac{-1}{\sqrt{\Delta_i}} + x_i \frac{\left(\frac{x_i + y_i}{2} - \xi^2 y_i \right)}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2)} - y_i \frac{\left(-\frac{x_i + y_i}{2} + \xi^2 x_i \right)}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2)} \right) \\ &= \imath \sum_i s_i J_i \left(\frac{-2\xi^2(1 - \xi^2) + \frac{(x_i + y_i)^2}{2} - 2\xi^2 x_i y_i}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2)} \right) \\ &= \imath \sum_i s_i J_i \left(-2 \frac{\Delta_i}{2\sqrt{\Delta_i} \xi^2 (1 - \xi^2)} \right) = -\frac{\imath}{\xi^2(1 - \xi^2)} \sum_i s_i J_i \sqrt{\Delta_i}. \end{aligned} \quad (\text{D.51})$$

D.2 The saddle point equation

We will use in the sequel the short hand notation $A^{\vec{B}} := \vec{A} \cdot \vec{B}$ for all vectors \vec{A} and \vec{B} . Squaring twice the saddle point eq. (69) we obtain, for all signs s_i ,

$$\left[J_3^2 \Delta_3 - J_1^2 \Delta_1 - J_2^2 \Delta_2 \right]^2 = 4 J_1^2 J_2^2 \Delta_1 \Delta_2 . \quad (\text{D.52})$$

We first translate eq. (D.52) in terms of angular momentum vectors

$$J_i^2 \Delta_i = (1 - \xi^2) \xi^2 J_i^2 + \xi^2 J_i^{\vec{n}} J_i^{\vec{k}} - \frac{1}{4} (J_i^{\vec{n}+\vec{k}})^2 . \quad (\text{D.53})$$

Then $J_3^2 \Delta_3 - J_1^2 \Delta_1 - J_2^2 \Delta_2$ computes to

$$\begin{aligned} & (1 - \xi^2) \xi^2 \left[J_3^2 - J_1^2 - J_2^2 \right] + \xi^2 \left[J_3^{\vec{n}} J_3^{\vec{k}} - J_1^{\vec{n}} J_1^{\vec{k}} - J_2^{\vec{n}} J_2^{\vec{k}} \right] \\ & - \frac{1}{4} \left[(J_3^{\vec{n}+\vec{k}})^2 - (J_1^{\vec{n}+\vec{k}})^2 - (J_2^{\vec{n}+\vec{k}})^2 \right] , \end{aligned} \quad (\text{D.54})$$

and using $\vec{J}_3 = -\vec{J}_1 - \vec{J}_2$, eq. (D.52) becomes

$$\begin{aligned} & \left\{ 2(1 - \xi^2) \xi^2 \vec{J}_1 \cdot \vec{J}_2 + \xi^2 (J_1^{\vec{n}} J_2^{\vec{k}} + J_2^{\vec{n}} J_1^{\vec{k}}) - \frac{1}{2} J_1^{\vec{n}+\vec{k}} J_2^{\vec{n}+\vec{k}} \right\}^2 \\ & = \left[2(1 - \xi^2) \xi^2 J_1^2 + 2\xi^2 J_1^{\vec{n}} J_1^{\vec{k}} - \frac{1}{2} (J_1^{\vec{n}+\vec{k}})^2 \right] \\ & \times \left[2(1 - \xi^2) \xi^2 J_2^2 + 2\xi^2 J_2^{\vec{n}} J_2^{\vec{k}} - \frac{1}{2} (J_2^{\vec{n}+\vec{k}})^2 \right] . \end{aligned} \quad (\text{D.55})$$

Collecting all terms on the LHS we get

$$\begin{aligned} & 4(1 - \xi^2)^2 \xi^4 \left[J_1^2 J_2^2 - (\vec{J}_1 \cdot \vec{J}_2)^2 \right] \\ & + 4(1 - \xi^2) \xi^4 \left[J_1^2 J_2^{\vec{n}} J_2^{\vec{k}} + J_2^2 J_1^{\vec{n}} J_1^{\vec{k}} - \vec{J}_1 \cdot \vec{J}_2 (J_1^{\vec{n}} J_2^{\vec{k}} + J_2^{\vec{n}} J_1^{\vec{k}}) \right] \\ & - (1 - \xi^2) \xi^2 \left[J_1^2 (J_2^{\vec{n}+\vec{k}})^2 + J_2^2 (J_1^{\vec{n}+\vec{k}})^2 - 2 \vec{J}_1 \cdot \vec{J}_2 J_1^{\vec{n}+\vec{k}} J_2^{\vec{n}+\vec{k}} \right] \\ & + \xi^4 \left[4 J_1^{\vec{n}} J_1^{\vec{k}} J_2^{\vec{n}} J_2^{\vec{k}} - (J_1^{\vec{n}} J_2^{\vec{k}} + J_2^{\vec{n}} J_1^{\vec{k}})^2 \right] \\ & - \xi^2 \left[J_1^{\vec{n}} J_1^{\vec{k}} (J_2^{\vec{n}+\vec{k}})^2 + J_2^{\vec{n}} J_2^{\vec{k}} (J_1^{\vec{n}+\vec{k}})^2 - (J_1^{\vec{n}} J_2^{\vec{k}} + J_2^{\vec{n}} J_1^{\vec{k}}) J_1^{\vec{n}+\vec{k}} J_2^{\vec{n}+\vec{k}} \right] = 0 . \end{aligned} \quad (\text{D.56})$$

Eq. (D.56) rewrites

$$\begin{aligned} & 4(1 - \xi^2)^2 \xi^4 \left[\vec{J}_1 \wedge \vec{J}_2 \right]^2 + 4(1 - \xi^2) \xi^4 \left[\vec{n} \wedge (\vec{J}_1 \wedge \vec{J}_2) \right] \cdot \left[\vec{k} \wedge (\vec{J}_1 \wedge \vec{J}_2) \right] \\ & - (1 - \xi^2) \xi^2 \left[(\vec{n} + \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2) \right]^2 - \xi^4 \left[(\vec{n} \wedge \vec{k}) \cdot (\vec{J}_1 \wedge \vec{J}_2) \right]^2 \\ & - \xi^2 \left[\vec{n} \cdot [(\vec{n} + \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2)] \right] \left[\vec{k} \cdot [(\vec{n} + \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2)] \right] = 0 . \end{aligned} \quad (\text{D.57})$$

Using $\vec{S} = \vec{J}_1 \wedge \vec{J}_2$, twice the oriented area of the triangle $\{\vec{J}_i\}$, the saddle point equation becomes

$$\begin{aligned} 0 & = 4(1 - \xi^2)^2 \xi^4 S^2 + 4(1 - \xi^2) \xi^4 \left[\vec{n} \wedge \vec{S} \right] \cdot \left[\vec{k} \wedge \vec{S} \right] \\ & - (1 - \xi^2) \xi^2 \left[(\vec{n} + \vec{k}) \wedge \vec{S} \right]^2 - \xi^4 \left[(\vec{n} \wedge \vec{k}) \cdot \vec{S} \right]^2 - \xi^2 \left[\vec{S} \cdot (\vec{n} \wedge \vec{k}) \right] \left[\vec{S} \cdot (\vec{k} \wedge \vec{n}) \right] , \end{aligned} \quad (\text{D.58})$$

and dividing by $(1 - \xi^2)\xi^2$ we obtain

$$0 = 4(1 - \xi^2)\xi^2 S^2 + 4\xi^2 [\vec{n} \wedge \vec{S}] \cdot [\vec{k} \wedge \vec{S}] + [\vec{S} \cdot (\vec{n} \wedge \vec{k})]^2 - [(\vec{n} + \vec{k}) \wedge \vec{S}]^2, \quad (\text{D.59})$$

that is

$$\begin{aligned} 0 &= 4\xi^4 S^2 - 4\xi^2 [S^2 + (\vec{n} \cdot \vec{k})S^2 - S^{\vec{n}} S^{\vec{k}}] \\ &\quad - S^2 (\vec{n} \wedge \vec{k})^2 + [\vec{S} \wedge (\vec{n} \wedge \vec{k})]^2 + S^2 (\vec{n} + \vec{k})^2 - (S^{\vec{n}} + S^{\vec{k}})^2. \end{aligned} \quad (\text{D.60})$$

The last line in eq. (D.60) computes

$$\begin{aligned} &-S^2 + S^2 (\vec{n} \cdot \vec{k})^2 + (S^{\vec{n}})^2 + (S^{\vec{k}})^2 - 2(\vec{n} \cdot \vec{k}) S^{\vec{n}} S^{\vec{k}} \\ &+ 2S^2 + 2S^2 (\vec{n} \cdot \vec{k}) - (S^{\vec{n}})^2 - (S^{\vec{k}})^2 - 2S^{\vec{n}} S^{\vec{k}} \\ &= [1 + (\vec{n} \cdot \vec{k})]^2 S^2 - 2[1 + (\vec{n} \cdot \vec{k})] S^{\vec{n}} S^{\vec{k}}, \end{aligned} \quad (\text{D.61})$$

and eq. (70) follows.

D.3 Evaluation of $J_i^2 \Delta_i^\pm$

Recall that $J_i^2 \Delta_i$ is

$$J_i^2 \Delta_i = (1 - \xi^2)\xi^2 J_i^2 + \xi^2 J_i^{\vec{n}} J_i^{\vec{k}} - \frac{1}{4} (J_i^{\vec{n}+\vec{k}})^2. \quad (\text{D.62})$$

Evaluated for $\xi_+^2 = \frac{1+(\vec{n} \cdot \vec{k})}{2}$, eq. (D.62) becomes

$$J_i^2 \Delta_i^+ = \frac{1 - (\vec{n} \cdot \vec{k})^2}{4} J_i^2 + \frac{1 + (\vec{n} \cdot \vec{k})}{2} J_i^{\vec{n}} J_i^{\vec{k}} - \frac{1}{4} (J_i^{\vec{n}} + J_i^{\vec{k}})^2, \quad (\text{D.63})$$

which further simplifies to

$$\begin{aligned} J_i^2 \Delta_i^+ &= \frac{1}{4} \left\{ (\vec{n} \wedge \vec{k})^2 J_i^2 + 2(\vec{n} \cdot \vec{k}) J_i^{\vec{n}} J_i^{\vec{k}} - (J_i^{\vec{n}})^2 - (J_i^{\vec{k}})^2 \right\} \\ &= \frac{1}{4} \left[(\vec{n} \wedge \vec{k})^2 J_i^2 + J_i^{\vec{n}} [(\vec{n} \wedge \vec{k}) \cdot (\vec{k} \wedge \vec{J}_i)] - J_i^{\vec{k}} [(\vec{n} \wedge \vec{k}) \cdot (\vec{n} \wedge \vec{J}_i)] \right], \end{aligned} \quad (\text{D.64})$$

and combining the last two terms this is

$$\begin{aligned} &\frac{1}{4} \left\{ (\vec{n} \wedge \vec{k})^2 J_i^2 + (\vec{n} \wedge \vec{k}) \cdot \left[(\vec{J}_i \wedge (\vec{k} \wedge \vec{n})) \wedge \vec{J}_i \right] \right\} \\ &= \frac{1}{4} \left\{ (\vec{n} \wedge \vec{k})^2 J_i^2 + (\vec{n} \wedge \vec{k}) \cdot \left[\vec{J}_i (\vec{J}_i \cdot (\vec{n} \wedge \vec{k})) - (\vec{n} \wedge \vec{k}) \vec{J}_i^2 \right] \right\}, \end{aligned} \quad (\text{D.65})$$

hence for ξ_+^2 we get

$$J_i \Delta_i^+ = \frac{1}{4} \left[\vec{J}_i \cdot (\vec{n} \wedge \vec{k}) \right]^2. \quad (\text{D.66})$$

Evaluated in $\xi_-^2 = \frac{1+(\vec{n} \cdot \vec{k})}{2} - \frac{S^{\vec{n}} S^{\vec{k}}}{S^2}$, $J_i^2 \Delta_i$ writes

$$\begin{aligned} J_i \Delta_i^- &= \left(\frac{1 - (\vec{n} \cdot \vec{k})}{2} + \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right) \left(\frac{1 + (\vec{n} \cdot \vec{k})}{2} - \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right) J_i^2 \\ &\quad + \left(\frac{1 + (\vec{n} \cdot \vec{k})}{2} - \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right) J_i^{\vec{n}} J_i^{\vec{k}} - \frac{1}{4} (J_i^{\vec{n}+\vec{k}})^2. \end{aligned} \quad (\text{D.67})$$

Combining all the terms common to the RHS in eq. (D.63) and eq. (D.67), we get

$$\begin{aligned} J_i \Delta_i^- &= \frac{1}{4} \left[\vec{J}_i \cdot (\vec{n} \wedge \vec{k}) \right]^2 + \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \left[(\vec{n} \cdot \vec{k}) J_i^2 - J_i^{\vec{n}} J_i^{\vec{k}} \right] - J_i^2 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^4} \\ &= \frac{1}{4S^2} \left\{ \left[\vec{S} (\vec{J}_i \cdot (\vec{n} \wedge \vec{k})) \right]^2 + 4S^{\vec{n}} S^{\vec{k}} \left[(\vec{n} \wedge \vec{J}_i) \cdot (\vec{k} \wedge \vec{J}_i) \right] - 4J_i^2 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^2} \right\}. \end{aligned} \quad (\text{D.68})$$

But note that $\vec{S} \cdot \vec{J}_i = 0$, hence the first term on the RHS above can be written as a double vector product, that is

$$\begin{aligned} J_i \Delta_i^- &= \frac{1}{4S^2} \left\{ \left[\vec{J}_i \wedge (\vec{S} \wedge (\vec{n} \wedge \vec{k})) \right]^2 + 4S^{\vec{n}} S^{\vec{k}} \left[(\vec{n} \wedge \vec{J}_i) \cdot (\vec{k} \wedge \vec{J}_i) \right] - 4J_i^2 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^2} \right\} \\ &= \frac{1}{4S^2} \left\{ \left[\vec{J}_i \wedge (\vec{n} S^{\vec{k}} + \vec{k} S^{\vec{n}}) \right]^2 - 4J_i^2 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^2} \right\} \\ &= \frac{1}{4S^4} \left\{ S^2 \left[\vec{J}_i \wedge (\vec{n} S^{\vec{k}} + \vec{k} S^{\vec{n}}) \right]^2 - 4J_i^2 (S^{\vec{n}} S^{\vec{k}})^2 \right\}. \end{aligned} \quad (\text{D.69})$$

And, as $A^2 B^2 = (\vec{A} \cdot \vec{B})^2 + (\vec{A} \wedge \vec{B})^2$, we have

$$\begin{aligned} J_i \Delta_i^- &= \frac{1}{4S^4} \left\{ \left[\vec{S} \cdot \left[\vec{J}_i \wedge (\vec{n} S^{\vec{k}} + \vec{k} S^{\vec{n}}) \right] \right]^2 + \left[\vec{J}_i (S^{\vec{n}} S^{\vec{k}} + S^{\vec{k}} S^{\vec{n}}) \right]^2 - 4J_i^2 (S^{\vec{n}} S^{\vec{k}})^2 \right\} \\ &= \frac{\left\{ \vec{J}_i \cdot [(\vec{n} \wedge \vec{S}) S^{\vec{k}} + (\vec{k} \wedge \vec{S}) S^{\vec{n}}] \right\}^2}{4S^4} = \frac{\left\{ \vec{J}_i \cdot [(\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}}] \right\}^2}{4S^4}. \end{aligned} \quad (\text{D.70})$$

D.4 Second derivative

Using $J_i \sqrt{\Delta_i^+}$ from eq. (74) and ξ_+^2 , we have

$$\begin{aligned} \epsilon_2^+ \epsilon_3^+ J_2 \sqrt{\Delta_2^+} J_3 \sqrt{\Delta_3^+} \left[(2\xi_+^2 - 1) J_1^2 - J_1^{\vec{n}} J_1^{\vec{k}} \right] + \odot_{123} &= \\ &= \frac{1}{4} \left\{ J_2^{\vec{n} \wedge \vec{k}} J_3^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_1)] + J_3^{\vec{n} \wedge \vec{k}} J_1^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_2)] \right. \\ &\quad \left. + J_1^{\vec{n} \wedge \vec{k}} J_2^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_3) \cdot (\vec{k} \wedge \vec{J}_3)] \right\}. \end{aligned} \quad (\text{D.71})$$

Substituting in the equation above $\vec{J}_3 = -\vec{J}_1 - \vec{J}_2$, the RHS writes

$$\begin{aligned} &\frac{1}{4} \left\{ -J_2^{\vec{n} \wedge \vec{k}} J_2^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_1)] - J_2^{\vec{n} \wedge \vec{k}} J_1^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_1)] \right. \\ &\quad - J_1^{\vec{n} \wedge \vec{k}} J_1^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_2)] - J_2^{\vec{n} \wedge \vec{k}} J_1^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_2)] \\ &\quad \left. + J_1^{\vec{n} \wedge \vec{k}} J_2^{\vec{n} \wedge \vec{k}} [(\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_1) + (\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_2)] \right\} \end{aligned}$$

$$+(\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_1) + (\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_2) \Big] \Big\} , \quad (\text{D.72})$$

canceling the appropriate cross terms, the remaining expression factors as

$$\begin{aligned} & -\frac{1}{4} \left\{ \left[J_2^{\vec{n} \wedge \vec{k}} (\vec{n} \wedge \vec{J}_1) - J_1^{\vec{n} \wedge \vec{k}} (\vec{n} \wedge \vec{J}_2) \right] \cdot \left[J_2^{\vec{n} \wedge \vec{k}} (\vec{k} \wedge \vec{J}_1) - J_1^{\vec{n} \wedge \vec{k}} (\vec{k} \wedge \vec{J}_2) \right] \right\} \\ & = -\frac{1}{4} \left\{ \vec{n} \wedge \left((\vec{n} \wedge \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2) \right) \right\} \cdot \left\{ \vec{k} \wedge \left((\vec{n} \wedge \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2) \right) \right\} , \end{aligned} \quad (\text{D.73})$$

developing the double vector products and taking into account that $\vec{n} \cdot (\vec{n} \wedge \vec{k}) = \vec{k} \cdot (\vec{n} \wedge \vec{k}) = 0$, we conclude

$$\epsilon_2^+ \epsilon_3^+ J_2 \sqrt{\Delta_2^+} J_3 \sqrt{\Delta_3^+} \left[(2\xi_+^2 - 1) J_1^2 - J_1^{\vec{n}} J_1^{\vec{k}} \right] + \odot_{123} = -\frac{1}{4} S^{\vec{n}} S^{\vec{k}} (\vec{n} \wedge \vec{k})^2 . \quad (\text{D.74})$$

For the ξ_-^2 root we have

$$\begin{aligned} & \epsilon_2^- \epsilon_3^- J_2 \sqrt{\Delta_2} J_3 \sqrt{\Delta_3} \left[(2\xi_-^2 - 1) J_1^2 - (\vec{n} \cdot \vec{J}_1) (\vec{k} \cdot \vec{J}_1) \right] + \odot_{123} = \\ & = \frac{1}{4} \left[\frac{J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}}}{S^2} \right] \left[\frac{J_3^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_3^{\vec{S} \wedge \vec{k}} S^{\vec{n}}}{S^2} \right] \left[(\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_1) - 2J_1^2 \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right] \\ & + \frac{1}{4} \left[\frac{J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}}}{S^2} \right] \left[\frac{J_3^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_3^{\vec{S} \wedge \vec{k}} S^{\vec{n}}}{S^2} \right] \left[(\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_2) - 2J_2^2 \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right] \\ & + \frac{1}{4} \left[\frac{J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}}}{S^2} \right] \left[\frac{J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}}}{S^2} \right] \left[(\vec{n} \wedge \vec{J}_3) \cdot (\vec{k} \wedge \vec{J}_3) - 2J_3^2 \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right] . \end{aligned} \quad (\text{D.75})$$

We substitute again in the equation above $\vec{J}_3 = -\vec{J}_1 - \vec{J}_2$. The coefficient of $\frac{1}{4S^2}$ computes, canceling the appropriate cross terms,

$$\begin{aligned} & - \left[J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right]^2 (\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_1) - \left[J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right]^2 (\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_2) \\ & + \left[J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right] \left[J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right] \\ & \times \left[(\vec{n} \wedge \vec{J}_1) \cdot (\vec{k} \wedge \vec{J}_2) + (\vec{n} \wedge \vec{J}_2) \cdot (\vec{k} \wedge \vec{J}_1) \right] , \end{aligned} \quad (\text{D.76})$$

while the coefficient of $-\frac{S^{\vec{k}} S^{\vec{n}}}{2S^6}$ is

$$\begin{aligned} & -J_1^2 \left[J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right]^2 - J_2^2 \left[J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right]^2 \\ & + 2\vec{J}_1 \cdot \vec{J}_2 \left[J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right] \left[J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right] . \end{aligned} \quad (\text{D.77})$$

The RHS of eq. (D.75) becomes

$$\begin{aligned} & -\frac{1}{4S^4} \left[\left(J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right) (\vec{n} \wedge \vec{J}_1) - \left(J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right) (\vec{n} \wedge \vec{J}_2) \right] \\ & \cdot \left[\left(J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right) (\vec{k} \wedge \vec{J}_1) - \left(J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right) (\vec{k} \wedge \vec{J}_2) \right] \end{aligned}$$

$$+\frac{S^{\vec{k}}S^{\vec{n}}}{2S^6} \left[\vec{J}_1 \left(J_2^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_2^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right) - \vec{J}_2 \left(J_1^{\vec{S} \wedge \vec{n}} S^{\vec{k}} + J_1^{\vec{S} \wedge \vec{k}} S^{\vec{n}} \right) \right]^2, \quad (\text{D.78})$$

which rewrites, combining the appropriate terms into double vector products as

$$\begin{aligned} & \frac{-1}{4S^4} \left\{ \vec{n} \wedge \left[(\vec{S} \wedge \vec{n}) \wedge (\vec{J}_1 \wedge \vec{J}_2) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2) S^{\vec{n}} \right] \right\} \\ & \cdot \left\{ \vec{k} \wedge \left[(\vec{S} \wedge \vec{n}) \wedge (\vec{J}_1 \wedge \vec{J}_2) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2) S^{\vec{n}} \right] \right\} \\ & + \frac{S^{\vec{k}}S^{\vec{n}}}{2S^6} \left[(\vec{S} \wedge \vec{n}) \wedge (\vec{J}_1 \wedge \vec{J}_2) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) \wedge (\vec{J}_1 \wedge \vec{J}_2) S^{\vec{n}} \right]^2, \end{aligned} \quad (\text{D.79})$$

and recalling that $\vec{J}_1 \wedge \vec{J}_2 = \vec{S}$ the above equation becomes

$$\begin{aligned} & \frac{-1}{4S^4} \left\{ \vec{n} \wedge \left[(\vec{S} \wedge \vec{n}) \wedge \vec{S} S^{\vec{k}} + (\vec{S} \wedge \vec{k}) \wedge \vec{S} S^{\vec{n}} \right] \right\} \\ & \cdot \left\{ \vec{k} \wedge \left[(\vec{S} \wedge \vec{n}) \wedge \vec{S} S^{\vec{k}} + (\vec{S} \wedge \vec{k}) \wedge \vec{S} S^{\vec{n}} \right] \right\} \\ & + \frac{S^{\vec{k}}S^{\vec{n}}}{2S^6} \left[(\vec{S} \wedge \vec{n}) \wedge \vec{S} S^{\vec{k}} + (\vec{S} \wedge \vec{k}) \wedge \vec{S} S^{\vec{n}} \right]^2. \end{aligned} \quad (\text{D.80})$$

We develop the double vector products in the first line and take into account $\vec{n} \cdot (\vec{S} \wedge \vec{n}) = \vec{k} \cdot (\vec{S} \wedge \vec{k}) = 0$. For the second line we use $(\vec{S} \wedge \vec{A})^2 = S^2 A^2 - (\vec{S} \cdot \vec{A})^2$ and $\vec{S} \cdot (\vec{S} \wedge \vec{n}) = \vec{S} \cdot (\vec{S} \wedge \vec{k}) = 0$ to rewrite the equation as

$$\begin{aligned} & \frac{-S^{\vec{n}}S^{\vec{k}}}{4S^4} \left\{ - \left[\vec{n} \cdot (\vec{S} \wedge \vec{k}) \right] \vec{S} + (\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}} \right\} \\ & \cdot \left\{ - \left[\vec{k} \cdot (\vec{S} \wedge \vec{n}) \right] \vec{S} + (\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}} \right\} \\ & + \frac{S^{\vec{k}}S^{\vec{n}}}{2S^4} \left[(\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}} \right]^2, \end{aligned} \quad (\text{D.81})$$

and noting that the cross term in the first scalar product cancel (again as $\vec{S} \cdot (\vec{S} \wedge \vec{n}) = \vec{S} \cdot (\vec{S} \wedge \vec{k}) = 0$), and combining the remaining three terms we get

$$\frac{S^{\vec{n}}S^{\vec{k}}}{4S^4} S^2 \left[\vec{S} \cdot (\vec{n} \wedge \vec{k}) \right]^2 + \frac{S^{\vec{k}}S^{\vec{n}}}{4S^4} \left[(\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}} \right]^2. \quad (\text{D.82})$$

Factoring \vec{S} in the second term and using $A^2 B^2 = (\vec{A} \cdot \vec{B})^2 + (\vec{A} \wedge \vec{B})^2$ this rewrites as

$$\frac{S^{\vec{n}}S^{\vec{k}}}{4} \left[(\vec{n} \wedge \vec{k})^2 - \frac{[\vec{n} \vec{S}^{\vec{k}} - \vec{k} \vec{S}^{\vec{n}}]^2}{S^2} + \frac{[\vec{n} S^{\vec{k}} + \vec{k} S^{\vec{n}}]^2}{S^2} - \frac{4(S^{\vec{n}}S^{\vec{k}})^2}{S^4} \right], \quad (\text{D.83})$$

thus we conclude

$$\begin{aligned} & \epsilon_2^- \epsilon_3^- J_2 \sqrt{\Delta_2} J_3 \sqrt{\Delta_3} \left[(2\xi_-^2 - 1) J_1^2 - (\vec{n} \cdot \vec{J}_1)(\vec{k} \cdot \vec{J}_1) \right] + \odot_{123} = \\ & = \frac{S^{\vec{n}}S^{\vec{k}}}{4} \left[(\vec{n} \wedge \vec{k})^2 + 4(\vec{n} \cdot \vec{k}) \frac{S^{\vec{n}}S^{\vec{k}}}{S^2} - \frac{4(S^{\vec{n}}S^{\vec{k}})^2}{S^4} \right]. \end{aligned} \quad (\text{D.84})$$

D.5 Function at the saddle points

We evaluate the relevant angles at the points ξ_{\pm}^2 by substituting eq. (72) and eq. (74) into eq. (26).

D.5.1 The angles ϕ_i^{\pm}

For the angles ϕ_i^{\pm} the direct substitution yields

$$\begin{aligned} \imath\epsilon_i^+ \phi_i^+ &= \ln \frac{(\vec{n} \wedge \vec{J}_i) \cdot (\vec{k} \wedge \vec{J}_i) + \imath J_i \left[\vec{J}_i \cdot (\vec{n} \wedge \vec{k}) \right]}{\sqrt{(\vec{n} \wedge \vec{J}_i)^2} \sqrt{(\vec{k} \wedge \vec{J}_i)^2}} \\ \imath\epsilon_i^- \phi_i^- &= \ln \frac{(\vec{n} \wedge \vec{J}_i) \cdot (\vec{k} \wedge \vec{J}_i) - 2J_i^2 \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} + \imath J_i \frac{\vec{J}_i \cdot \left[(\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}} \right]}{S^2}}{\sqrt{(\vec{n} \wedge \vec{J}_i)^2} \sqrt{(\vec{k} \wedge \vec{J}_i)^2}}. \end{aligned} \quad (\text{D.85})$$

Consider first the denominator of $\imath\phi_i^-$ multiplied by S^2 , namely

$$\begin{aligned} &S^2 (\vec{n} \wedge \vec{J}_i) \cdot (\vec{k} \wedge \vec{J}_i) - 2J_i^2 S^{\vec{n}} S^{\vec{k}} + \imath J_i \vec{J}_i \cdot \left[(\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}} \right] \\ &= [\vec{S} \wedge (\vec{n} \wedge \vec{J}_i)] \cdot [\vec{S} \wedge (\vec{k} \wedge \vec{J}_i)] + [\vec{S} \cdot (\vec{n} \wedge \vec{J}_i)] [\vec{S} \cdot (\vec{k} \wedge \vec{J}_i)] \\ &\quad - 2J_i^2 S^{\vec{n}} S^{\vec{k}} + \imath J_i \vec{J}_i \cdot \left[(\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}} \right] \\ &= \left[\vec{n} \cdot (\vec{J}_i \wedge \vec{S}) + \imath J_i S^{\vec{n}} \right] \left[\vec{k} \cdot (\vec{J}_i \wedge \vec{S}) + \imath J_i S^{\vec{k}} \right], \end{aligned} \quad (\text{D.86})$$

hence

$$\imath\epsilon_i^- \phi_i^- = \imath\Phi_{\vec{n}}^i + \imath\Phi_{\vec{k}}^i \quad \imath\Phi_{\vec{n}}^i = \ln \frac{\vec{n} \cdot (\vec{J}_i \wedge \vec{S}) + \imath J_i S^{\vec{n}}}{S \sqrt{(\vec{n} \wedge \vec{J}_i)^2}}. \quad (\text{D.87})$$

Note that

$$\begin{aligned} &\left[\vec{n} \cdot (\vec{J}_i \wedge \vec{S}) + \imath J_i S^{\vec{n}} \right] \left[\vec{k} \cdot (\vec{J}_i \wedge \vec{S}) - \imath J_i S^{\vec{k}} \right] = [\vec{S} \cdot (\vec{n} \wedge \vec{J}_i)] [\vec{S} \cdot (\vec{k} \wedge \vec{J}_i)] + J_i^2 S^{\vec{n}} S^{\vec{k}} \\ &\quad + \imath J_i \vec{J}_i \cdot \left[\vec{S} \wedge (\vec{k} S^{\vec{n}} - \vec{n} S^{\vec{k}}) \right] = S^2 (\vec{n} \wedge \vec{J}_i) \cdot (\vec{k} \wedge \vec{J}_i) - [\vec{S} \wedge (\vec{n} \wedge \vec{J}_i)] \cdot [\vec{S} \wedge (\vec{k} \wedge \vec{J}_i)] \\ &\quad + J_i^2 S^{\vec{n}} S^{\vec{k}} + \imath J_i \vec{J}_i \cdot \left[\vec{S} \wedge (\vec{S} \wedge (\vec{k} \wedge \vec{n})) \right], \end{aligned} \quad (\text{D.88})$$

and developing the double vector products, taking into account $\vec{S} \cdot \vec{J}_i = 0$, we conclude

$$\imath\epsilon_i^+ \phi_i^+ = \imath\Phi_{\vec{n}}^i - \imath\Phi_{\vec{k}}^i. \quad (\text{D.89})$$

D.5.2 The angles $\epsilon_1^\pm \psi_1^\pm - \epsilon_3^\pm \psi_3^\pm$

We will denote in this section $\vec{A} \wedge \vec{B} = \vec{A}^{\wedge \vec{B}}$ Direct substitution of ξ_+^2 and ξ_-^2 yields

$$\begin{aligned} \imath \epsilon_i^+ \psi_i^+ &= \ln \frac{J_i^{\vec{k}} - J_i^{\vec{n}}(\vec{n} \cdot \vec{k}) + \imath \vec{J}_i \cdot (\vec{n} \wedge \vec{k})}{\sqrt{[1 - (\vec{n} \cdot \vec{k})^2](\vec{n} \wedge \vec{J}_i)^2}} \\ \imath \epsilon_i^- \psi_i^- &= \ln \frac{J_i^{\vec{k}} - J_i^{\vec{n}}(\vec{n} \cdot \vec{k}) + 2J_i^{\vec{n}} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} + \imath \frac{\vec{J}_i \cdot [(\vec{S} \wedge \vec{n}) S^{\vec{k}} + (\vec{S} \wedge \vec{k}) S^{\vec{n}}]}{S^2}}{\sqrt{[1 - (\vec{n} \cdot \vec{k})^2 + 4(\vec{n} \cdot \vec{k}) \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} - 4 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^4}](\vec{n} \wedge \vec{J}_i)^2}}. \end{aligned} \quad (\text{D.90})$$

To evaluate $\imath \epsilon_1^+ \psi_1^+ - \imath \epsilon_3^+ \psi_3^+$ we take apart the numerator

$$\begin{aligned} & \left[J_1^{\vec{n} \wedge (\vec{k} \wedge \vec{n})} + \imath J_1^{\vec{n} \wedge \vec{k}} \right] \left[J_3^{\vec{n} \wedge (\vec{k} \wedge \vec{n})} - \imath J_3^{\vec{n} \wedge \vec{k}} \right] \\ &= -\vec{J}_1^{\wedge [\vec{n} \wedge (\vec{k} \wedge \vec{n})]} \cdot \vec{J}_3^{\wedge [\vec{n} \wedge (\vec{k} \wedge \vec{n})]} + \vec{J}_1 \cdot \vec{J}_3 \left(\vec{n} \wedge (\vec{k} \wedge \vec{n}) \right)^2 + J_1^{\vec{n} \wedge \vec{k}} J_3^{\vec{n} \wedge \vec{k}} \\ &+ \imath \left(J_1^{\vec{n} \wedge \vec{k}} J_3^{\vec{n} \wedge (\vec{k} \wedge \vec{n})} - J_1^{\vec{n} \wedge (\vec{k} \wedge \vec{n})} J_3^{\vec{n} \wedge \vec{k}} \right). \end{aligned} \quad (\text{D.91})$$

Taking into account $\vec{n} \cdot (\vec{k} \wedge \vec{n}) = 0$ this writes

$$\begin{aligned} & -(\vec{k} \wedge \vec{n})^2 J_1^{\vec{n}} J_3^{\vec{n}} + \vec{J}_1 \cdot \vec{J}_3 (\vec{k} \wedge \vec{n})^2 + \imath \vec{J}_1 \cdot \left[\vec{J}_3 \wedge [(\vec{n} \wedge \vec{k}) \wedge (\vec{n} \wedge (\vec{k} \wedge \vec{n}))] \right] \\ &= (\vec{k} \wedge \vec{n})^2 (\vec{n} \wedge \vec{J}_1) \cdot (\vec{n} \wedge \vec{J}_3) - \imath \vec{J}_1 \cdot \left[\vec{J}_3 \wedge [\vec{n}(\vec{n} \wedge \vec{k})^2] \right], \end{aligned} \quad (\text{D.92})$$

hence

$$\imath \epsilon_1^+ \psi_1^+ - \imath \epsilon_3^+ \psi_3^+ = \ln \frac{(\vec{n} \wedge \vec{J}_1) \cdot (\vec{n} \wedge \vec{J}_3) - \imath \vec{n} \cdot (\vec{J}_1 \wedge \vec{J}_3)}{\sqrt{(\vec{n} \wedge \vec{J}_1)^2 (\vec{n} \wedge \vec{J}_3)^2}} = \imath \Psi_{\vec{n}}^{13}. \quad (\text{D.93})$$

To evaluate $\imath \epsilon_1^- \psi_1^- - \imath \epsilon_3^- \psi_3^-$ we again take apart the numerator

$$\left(J_1^{\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2}} + \imath J_1^{\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}} \right) \left(J_3^{\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2}} - \imath J_3^{\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}} \right). \quad (\text{D.94})$$

The real part is

$$\begin{aligned} & J_1^{\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2}} J_3^{\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2}} + J_1^{\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}} J_3^{\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}} \\ &= -\vec{J}_1^{\wedge [\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2}]} \cdot \vec{J}_3^{\wedge [\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2}]} + \vec{J}_1 \cdot \vec{J}_3 \left(\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right)^2 \\ &- \vec{J}_1^{\wedge [\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}]} \cdot \vec{J}_3^{\wedge [\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}]} + \vec{J}_1 \cdot \vec{J}_3 \left(\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2} \right)^2, \end{aligned} \quad (\text{D.95})$$

which rewrites, taking into account $\vec{S} \cdot \vec{J}_i = 0$,

$$-\left(\vec{n} J_1^{\vec{k} \wedge \vec{n}} - (\vec{k} \wedge \vec{n}) J_1^{\vec{n}} + 2\vec{J}_1 \wedge \vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right) \cdot \left(\vec{n} J_3^{\vec{k} \wedge \vec{n}} - (\vec{k} \wedge \vec{n}) J_3^{\vec{n}} + 2\vec{J}_3 \wedge \vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right)$$

$$\begin{aligned}
& -\vec{S}\left(J_1^{\vec{n}}\frac{S^{\vec{k}}}{S^2} + J_1^{\vec{k}}\frac{S^{\vec{n}}}{S^2}\right) \cdot \vec{S}\left(J_3^{\vec{n}}\frac{S^{\vec{k}}}{S^2} + J_3^{\vec{k}}\frac{S^{\vec{n}}}{S^2}\right) \\
& + \vec{J}_1 \cdot \vec{J}_3 \left[(\vec{n} \wedge (\vec{k} \wedge \vec{n}))^2 + 4\frac{(S^{\vec{n}}S^{\vec{k}})^2}{S^4} + S^2\frac{(\vec{n}S^{\vec{k}} + \vec{k}S^{\vec{n}})^2}{S^4} - 4\frac{(S^{\vec{n}}S^{\vec{k}})^2}{S^4} \right].
\end{aligned} \tag{D.96}$$

Developing the products in the first line we get

$$\begin{aligned}
& -J_1^{\vec{k} \wedge \vec{n}} J_3^{\vec{k} \wedge \vec{n}} - (\vec{n} \wedge \vec{k})^2 J_1^{\vec{n}} J_3^{\vec{n}} - 4\frac{(S^{\vec{n}}S^{\vec{k}})^2}{S^2} (\vec{J}_1 \wedge \vec{n}) \cdot (\vec{J}_3 \wedge \vec{n}) \\
& + 2\frac{S^{\vec{n}}S^{\vec{k}}}{S^2} \left[(\vec{J}_1 \wedge \vec{n}) \cdot (\vec{k} \wedge \vec{n}) J_3^{\vec{n}} + (\vec{J}_3 \wedge \vec{n}) \cdot (\vec{k} \wedge \vec{n}) J_1^{\vec{n}} \right] \\
& - \frac{1}{S^2} \left(J_1^{\vec{n}} S^{\vec{k}} + J_1^{\vec{k}} S^{\vec{n}} \right) \left(J_3^{\vec{n}} S^{\vec{k}} + J_3^{\vec{k}} S^{\vec{n}} \right) + \vec{J}_1 \cdot \vec{J}_3 \left((\vec{n} \wedge \vec{k})^2 + \frac{(\vec{n}S^{\vec{k}} + \vec{k}S^{\vec{n}})^2}{S^2} \right),
\end{aligned} \tag{D.97}$$

which is, expanding the second line,

$$\begin{aligned}
& (\vec{J}_1 \wedge \vec{n}) \cdot (\vec{J}_3 \wedge \vec{n}) \left((\vec{n} \wedge \vec{k})^2 - 4\frac{(S^{\vec{n}}S^{\vec{k}})^2}{S^2} \right) - J_1^{\vec{k} \wedge \vec{n}} J_3^{\vec{k} \wedge \vec{n}} \\
& + 2\frac{S^{\vec{n}}S^{\vec{k}}}{S^2} \left[J_1^{\vec{k}} J_3^{\vec{n}} + J_1^{\vec{n}} J_3^{\vec{k}} - 2(\vec{n} \cdot \vec{k}) J_1^{\vec{n}} J_3^{\vec{n}} \right] - \frac{1}{S^2} \left(J_1^{\vec{n}} S^{\vec{k}} + J_1^{\vec{k}} S^{\vec{n}} \right) \left(J_3^{\vec{n}} S^{\vec{k}} + J_3^{\vec{k}} S^{\vec{n}} \right) \\
& + \vec{J}_1 \cdot \vec{J}_3 \frac{(\vec{n}S^{\vec{k}} + \vec{k}S^{\vec{n}})^2}{S^2}.
\end{aligned} \tag{D.98}$$

Combining the cross terms in the second line and using $J_1^{\vec{k} \wedge \vec{n}} J_3^{\vec{k} \wedge \vec{n}} = J_1^{\vec{n} \wedge \vec{k}} J_3^{\vec{n} \wedge \vec{k}}$ we obtain

$$\begin{aligned}
& (\vec{J}_1 \wedge \vec{n}) \cdot (\vec{J}_3 \wedge \vec{n}) \left((\vec{n} \wedge \vec{k})^2 - 4\frac{(S^{\vec{n}}S^{\vec{k}})^2}{S^2} \right) - 4(\vec{n} \cdot \vec{k}) J_1^{\vec{n}} J_3^{\vec{n}} \frac{S^{\vec{n}}S^{\vec{k}}}{S^2} \\
& - J_1^{\vec{n} \wedge \vec{k}} J_3^{\vec{n} \wedge \vec{k}} - \frac{1}{S^2} J_1^{\vec{S} \wedge (\vec{n} \wedge \vec{k})} J_3^{\vec{S} \wedge (\vec{n} \wedge \vec{k})} + \vec{J}_1 \cdot \vec{J}_3 \frac{(\vec{n}S^{\vec{k}} + \vec{k}S^{\vec{n}})^2}{S^2},
\end{aligned} \tag{D.99}$$

and computing the middle term on the second line taking into account $\vec{S} \cdot \vec{J}_i = 0$, we obtain

$$\begin{aligned}
& (\vec{J}_1 \wedge \vec{n}) \cdot (\vec{J}_3 \wedge \vec{n}) \left((\vec{n} \wedge \vec{k})^2 - 4\frac{(S^{\vec{n}}S^{\vec{k}})^2}{S^2} \right) - 4(\vec{n} \cdot \vec{k}) J_1^{\vec{n}} J_3^{\vec{n}} \frac{S^{\vec{n}}S^{\vec{k}}}{S^2} \\
& + \vec{J}_1 \cdot \vec{J}_3 \frac{(\vec{n}S^{\vec{k}} + \vec{k}S^{\vec{n}})^2 - (\vec{n}S^{\vec{k}} - \vec{k}S^{\vec{n}})^2}{S^2},
\end{aligned} \tag{D.100}$$

hence the real part is

$$(\vec{J}_1 \wedge \vec{n}) \cdot (\vec{J}_3 \wedge \vec{n}) \left[(\vec{n} \wedge \vec{k})^2 - 4\frac{(S^{\vec{n}}S^{\vec{k}})^2}{S^2} + 4\frac{S^{\vec{n}}S^{\vec{k}}(\vec{n} \cdot \vec{k})}{S^2} \right]. \tag{D.101}$$

The imaginary part of the numerator (D.94) writes

$$J_1^{\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}} J_3^{\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}}S^{\vec{k}}}{S^2}} - J_1^{\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}}S^{\vec{k}}}{S^2}} J_3^{\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2}}. \tag{D.102}$$

We start by expressing it as

$$-\vec{J}_1^\wedge \left[\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}}S^{\vec{k}}}{S^2} \right] \cdot \vec{J}_3^\wedge \left[\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2} \right]$$

$$+\vec{J}_3 \wedge \left[\vec{n} \wedge (\vec{k} \wedge \vec{n}) + 2\vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right] \cdot \vec{J}_1 \wedge \left[\vec{S} \wedge \vec{n} \frac{S^{\vec{k}}}{S^2} + \vec{S} \wedge \vec{k} \frac{S^{\vec{n}}}{S^2} \right], \quad (\text{D.103})$$

as the cross terms in the development of the two scalar products cancel. This computes further to

$$\begin{aligned} & - \left(\vec{n} J_1^{\vec{k} \wedge \vec{n}} - (\vec{k} \wedge \vec{n}) J_1^{\vec{n}} + 2\vec{J}_1 \wedge \vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right) \cdot \vec{S} \left(J_3^{\vec{n}} \frac{S^{\vec{k}}}{S^2} + J_3^{\vec{k}} \frac{S^{\vec{n}}}{S^2} \right) \\ & + \left(\vec{n} J_3^{\vec{k} \wedge \vec{n}} - (\vec{k} \wedge \vec{n}) J_3^{\vec{n}} + 2\vec{J}_3 \wedge \vec{n} \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right) \cdot \vec{S} \left(J_1^{\vec{n}} \frac{S^{\vec{k}}}{S^2} + J_1^{\vec{k}} \frac{S^{\vec{n}}}{S^2} \right). \end{aligned} \quad (\text{D.104})$$

Grouping together similar terms we get

$$\begin{aligned} & \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \left(\vec{J}_3 \wedge \vec{J}_1 \right) \cdot \left((\vec{k} \wedge \vec{n}) \wedge \vec{n} \right) + \frac{(S^{\vec{n}})^2}{S^2} \left(\vec{J}_3 \wedge \vec{J}_1 \right) \cdot \left((\vec{k} \wedge \vec{n}) \wedge \vec{k} \right) \\ & - [(\vec{k} \wedge \vec{n}) \cdot \vec{S}] \frac{S^{\vec{n}}}{S^2} \left(\vec{J}_3 \wedge \vec{J}_1 \right) \cdot (\vec{n} \wedge \vec{k}) \\ & + 2 \frac{S^{\vec{n}} (S^{\vec{k}})^2}{S^4} \left[\left(\vec{n} \wedge (\vec{J}_3 \wedge \vec{J}_1) \right) \wedge \vec{n} \right] \cdot \vec{S} + 2 \frac{(S^{\vec{n}})^2 S^{\vec{k}}}{S^4} \left[\left(\vec{k} \wedge (\vec{J}_3 \wedge \vec{J}_1) \right) \wedge \vec{n} \right] \cdot \vec{S}. \end{aligned} \quad (\text{D.105})$$

Recognizing $\vec{S} = \vec{J}_3 \wedge \vec{J}_1$, the above writes

$$\begin{aligned} & \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} [S^{\vec{n}} (\vec{n} \cdot \vec{k}) - S^{\vec{k}}] + \frac{(S^{\vec{n}})^2}{S^2} [S^{\vec{n}} - S^{\vec{k}} (\vec{n} \cdot \vec{k})] + \frac{S^{\vec{n}}}{S^2} [\vec{S} \cdot (\vec{n} \wedge \vec{k})]^2 \\ & + 2 \frac{S^{\vec{n}} (S^{\vec{k}})^2}{S^4} \left(- (S^{\vec{n}})^2 + S^2 \right) + 2 \frac{(S^{\vec{n}})^2 S^{\vec{k}}}{S^4} \left(- S^{\vec{n}} S^{\vec{k}} + (\vec{n} \cdot \vec{k}) S^2 \right) \\ & = S^{\vec{n}} \left[(\vec{n} \wedge \vec{k})^2 - \frac{(\vec{n} S^{\vec{k}} - \vec{k} S^{\vec{n}})^2}{S^2} - \frac{(S^{\vec{k}})^2}{S^2} + \frac{(S^{\vec{n}})^2}{S^2} - 4 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^4} + 2 \frac{(S^{\vec{k}})^2}{S^2} + 2(\vec{n} \cdot \vec{k}) \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} \right] \\ & = S^{\vec{n}} \left[(\vec{n} \wedge \vec{k})^2 + 4 \frac{S^{\vec{n}} S^{\vec{k}}}{S^2} - 4 \frac{(S^{\vec{n}} S^{\vec{k}})^2}{S^4} \right]. \end{aligned} \quad (\text{D.106})$$

In conclusion $\imath \epsilon_1^- \psi_1^- - \imath \epsilon_3^- \psi_3^-$ is

$$\imath \epsilon_1^- \psi_1^- - \imath \epsilon_3^- \psi_3^- = \ln \frac{(\vec{n} \wedge \vec{J}_1) \cdot (\vec{n} \wedge \vec{J}_3) + \imath \vec{n} \cdot (\vec{J}_3 \wedge \vec{J}_1)}{\sqrt{(\vec{n} \wedge \vec{J}_1)^2 (\vec{n} \wedge \vec{J}_3)^2}} = \imath \Psi_{\vec{n}}^{13}. \quad (\text{D.107})$$

Following similar manipulations we get

$$\imath \epsilon_2^\pm \psi_2^\pm - \imath \epsilon_3^\pm \psi_3^\pm = \ln \frac{(\vec{n} \wedge \vec{J}_2) \cdot (\vec{n} \wedge \vec{J}_3) + \imath \vec{n} \cdot (\vec{J}_3 \wedge \vec{J}_2)}{\sqrt{(\vec{n} \wedge \vec{J}_2)^2 (\vec{n} \wedge \vec{J}_3)^2}} = \imath \Psi_{\vec{n}}^{23}. \quad (\text{D.108})$$

For the angles ω_i , recall that $\omega_{\vec{n}, \vec{k}}$ can be written in terms of $\psi_{-\vec{k}, -\vec{n}}$. Note that due to the choice of the determination of the $\sqrt{\Delta_i^\pm}$ the correct relation is $\omega_{\vec{n}, \vec{k}}^+ = -\psi_{-\vec{k}, -\vec{n}}^+$ and $\omega_{\vec{n}, \vec{k}}^- = \psi_{-\vec{k}, -\vec{n}}^-$. Moreover, as $\Psi_{-\vec{k}} = -\Psi_{\vec{k}}$, we conclude

$$\imath \epsilon_1^\pm \omega_1^\pm - \imath \epsilon_3^\pm \omega_3^\pm = \pm \imath \Psi_{\vec{k}}^{13}, \quad \imath \epsilon_2^\pm \omega_2^\pm - \imath \epsilon_3^\pm \omega_3^\pm = \pm \imath \Psi_{\vec{k}}^{23}. \quad (\text{D.109})$$

E Boundary terms in the Euler Maclaurin formula

Using the short hand notation $F(t)$ for $F(J, M, M', t)$, the remainder terms in the EM formula write

$$- B_1 [F(t_{\max}) + F(t_{\min})] + \sum_k \frac{B_{2k}}{(2k)!} [F^{(2k-1)}(t_{\max}) - F^{(2k-1)}(t_{\min})] . \quad (\text{E.110})$$

In this section we deal with generic Wigner matrices, that is we consistently assume that $0 < \xi^2 < 1$. Note that

$$t_{\min} = \max\{0, M - M'\} , \quad t_{\max} = \min\{J + M, J - M'\} . \quad (\text{E.111})$$

For simplicity we will detail the diagonal matrix elements $M = M'$. By continuity the region in which our results apply extends to some strip $|M - M'| < P$. For such elements $t_{\min} = 0$ and, for $M > 0$, $t_{\max} = J - M$. The Stirling approximations become easily upper and lower bounds, at the price of some constants, thus by Appendix A we obtain

$$\frac{C^{\min}}{J} \sqrt{K(x, x, u)} e^{J\Re f(x, x, u)} < |F(t)| < \frac{C^{\max}}{J} \sqrt{K(x, x, u)} e^{J\Re f(x, x, u)} , \quad (\text{E.112})$$

with

$$\begin{aligned} f(x, x, u) = & -\imath(\alpha + \gamma)x + \imath\pi u + (1 - u) \ln \xi^2 + u \ln(1 - \xi^2) \\ & + (1 - x) \ln(1 - x) + (1 + x) \ln(1 + x) - 2u \ln u \\ & - (1 + x - u) \ln(1 + x - u) - (1 - x - u) \ln(1 - x - u) . \end{aligned} \quad (\text{E.113})$$

and

$$K(x, u) = \frac{(1 - x^2)}{(1 + x - u)(1 - x - u)u^2} . \quad (\text{E.114})$$

The behavior of the higher derivative terms in the EM formula is governed by $F^{(k)}(t_{\min})$ and $F^{(k)}(t_{\max})$. To see this, collect all factors depending on t in $F(t)$ and write

$$\begin{aligned} F(t) &= q(J, M) p_{J, M}(t) , \\ p_{J, M}(t) &= \frac{e^{At}}{\Gamma(J + M - t + 1) \Gamma(J - M - t + 1) [\Gamma(t + 1)]^2} , \\ A &:= \imath[\pi - 2 \ln \xi - 2 \ln \eta] . \end{aligned} \quad (\text{E.115})$$

Hence $F^{(k)} = q(J, M) p_{J, M}^{(k)}$, and the first derivatives expresses in terms of

$$\begin{aligned} \frac{d}{dt} p_{J, M}(t) &= p_{J, M}(t) \left\{ A + \frac{\Gamma'(J + M - t + 1)}{\Gamma(J + M - t + 1)} \right. \\ &\quad \left. + \frac{\Gamma'(J - M - t + 1)}{\Gamma(J - M - t + 1)} - 2 \frac{\Gamma'(t + 1)}{\Gamma(t + 1)} \right\} \\ &= p_{J, M}(t) \left\{ A + \psi^{(0)}(J + M - t + 1) \right. \\ &\quad \left. + \psi^{(0)}(J - M - t + 1) - 2\psi^{(0)}(t + 1) \right\} , \end{aligned} \quad (\text{E.116})$$

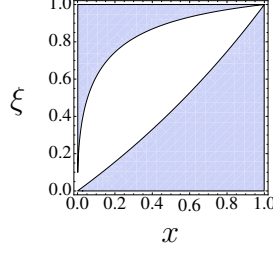


Figure 3: Shaded region where the EM corrections are exponentially suppressed

with $\psi^{(0)}(t)$ denoting the digamma function. For integer arguments

$$\psi^{(0)}(m+1) = -\gamma_0 + \sum_{k=1}^m \frac{1}{k}, \quad (\text{E.117})$$

hence $|F'(t)| < C \ln J |F(t)|$ for some constant C . Higher order derivatives of eq. (E.116) write in terms of higher order polygamma functions $\psi^{(n)} = d^n \psi^{(0)} / dt^n$. For all k , $\psi^{(2k)}(X) \leq \psi^{(0)}(X)$ at large X , therefore the k 'th derivative is dominated by

$$F^{(2k-1)}(t) = F(t) \left\{ \left[A + \sum_{i=1}^4 \pm \psi^{(0)}(X_i) \right]^{2k-1} + \dots \right\}. \quad (\text{E.118})$$

Then $|F^{(k)}| < C(\ln J)^k |F(t)|$ for some constant C .

From eq. (E.112) we conclude that both $|F(t_{\min})|$ and $|F(t_{\max})|$, as well as all their derivatives are a priori exponentially suppressed in the region where $\Re f(x, y, u_{\min}) < 0$ and $\Re f(x, y, u_{\max}) < 0$. As

$$\begin{aligned} \Re f(x, x, 0) &= \ln \xi^2 \\ \Re f(x, x, 1-x) &= x \ln \xi^2 + (1-x) \ln(1-\xi^2) \\ &\quad + (1+x) \ln(1+x) - (1-x) \ln(1-x) - 2x \ln(2x), \end{aligned} \quad (\text{E.119})$$

we conclude that the derivative corrections coming from $t_{\min} = 0$ are always suppressed term by term. However the situation is markedly different for the corrections coming from $t_{\max} = J - M$. At fixed ξ^2 , the corrections are exponentially suppressed for x close enough to either 0 or 1, but the maximum of $\Re f(x, x, 1-x)$, achieved for $x = \frac{\xi}{\sqrt{4-3\xi^2}}$ is $\ln \frac{(\xi + \sqrt{4-3\xi^2})^2}{4} > 0$, hence there exists some interval in which, term by term, the derivative terms are bounded from below by an exponential blow up. In this region our EM SPA approximation should a priori fail (see also figure 3).

A second set of EM derivative terms come when passing from eq. (36) to eq. (37), involving derivatives $\frac{\partial^n}{(\partial_x)^n} D_{xJ,xJ}^J|_{x=\pm 1}$. Using Appendix C, eq. (C.39) we note that all these derivatives yield some function times $D_{xJ,xJ}^J$. As

$$D_{-J,-J}^J(g) = \xi^{2J} e^{+i(\alpha+\gamma)J}, \quad D_{JJ}^J(g) = \xi^{2J} e^{-i(\alpha+\gamma)J}, \quad (\text{E.120})$$

all such derivative terms are exponentially suppressed for large J .

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